Pohlmeyer-reduced form of string theory in $AdS_5 \times S^5$: semiclassical expansion

2009 J. Phys. A: Math. Theor. 42 375204

(http://iopscience.iop.org/1751-8121/42/37/375204)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 375204 (34pp) doi:10.1088/1751-8113/42/37/375204

Pohlmeyer-reduced form of string theory in $AdS_5 \times S^5$: semiclassical expansion

B Hoare, Y Iwashita and A A Tseytlin[1]

Theoretical Physics Group, Blackett Laboratory, Imperial College, London SW7 2AZ, UK

E-mail: benjamin.hoare08@imperial.ac.uk, yukinori.iwashita07@imperial.ac.uk and tseytlin@imperial.ac.uk

Received 25 June 2009, in final form 27 June 2009
Published 25 August 2009
Online at stacks.iop.org/JPhysA/42/375204

Abstract

We consider the Pohlmeyer-reduced formulation of the $AdS_5 \times S^5$ superstring. It is constructed by introducing new variables which are algebraically related to supercoset current components so that the Virasoro conditions are automatically solved. The reduced theory is a gauged WZW model supplemented with an integrable potential and fermionic terms that ensure its UV finiteness. The original superstring theory and its reduced counterpart are closely related at the classical level, and we conjecture that they remain related at the quantum level as well, in the sense that their quantum partition functions evaluated on respective classical solutions are equal. We provide evidence for the validity of this conjecture at the one-loop level, i.e. at the first non-trivial order of the semiclassical expansion near several classes of classical solutions.

PACS numbers: 11.10.−z, 11.25.Tq

1. Introduction

In this paper we continue the investigation of the Pohlmeyer-reduced form of the $AdS_5 \times S^5$ superstring theory initiated in [1–4].

The original Pohlmeyer reduction procedure [5] relates the classical equations of motion of the sigma model on S^2 to the sine-Gordon equation. The reduction may be interpreted [6–8], as solving the Virasoro conditions in the classical conformal-gauge string theory on $\mathbb{R}_t \times S^2$ (with the residual conformal diffeomorphisms fixed by $t = \mu\tau$ condition) in terms of the remaining physical degree of freedom: identified as the angle variable of the sine-Gordon model. This relation between the S^2 sigma model and the sine-Gordon model (and its generalizations to other similar bosonic sigma models) was used for explicit construction

[1] Also at Lebedev Institute, Russian Academy of Sciences, Leninskiy prospekt 53, Moscow, Russia.

of several interesting classical string solutions on symmetric spaces such as S^n and AdS_n (see, e.g., [9–17]).

An attractive feature of the Pohlmeyer-reduced form of the string theory sigma model is that while it involves only the physical ('transverse') degrees of freedom it still has manifest 2D Lorentz invariance. It would be very useful to have such a formulation for the quantum $\mathrm{AdS}_5 \times S^5$ string theory.

Starting with the equations of motion of the $\mathrm{AdS}_5 \times S^5$ superstring described by the $\frac{F}{G} = \frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$ supercoset, which may be written in terms of the $PSU(2,2|4)$ current one may solve the Virasoro conditions by introducing the new variables $g \in G = Sp(2,2) \times Sp(4)$, $A_\pm \in \mathfrak{h}$,[2] and $\Psi_{L,R}$, which are algebraically related to the current components. The resulting equations can then be obtained from a local action $I_{\mathrm{red}}(g, A_\pm, \Psi_{L,R})$ which happens to be the G/H gauged WZW model modified by an H-invariant potential and supplemented by the 2D fermionic terms (see [1] and (2.24), (2.25) below). This action, which defines the reduced theory, is 2D Lorentz invariant and (after fixing the residual H gauge symmetry) involves only the physical number (8+8) of bosonic and fermionic degrees of freedom.

The original $\mathrm{AdS}_5 \times S^5$ superstring theory and the reduced theory are essentially equivalent at the classical level, having closely related integrable structures and sets of classical solutions. The question that we would like to address here is if this correspondence may extend to the quantum level.

Since the classical Pohlmeyer reduction utilizes conformal invariance, it has a chance to apply at the quantum level only if the sigma model one starts with is UV finite. This is the case for the $\mathrm{AdS}_5 \times S^5$ superstring sigma model [18–21], which is a combination of the AdS_5 and the S^5 sigma models 'glued' together by the Green–Schwarz fermions into a conformal 2D theory. For consistency, the corresponding reduced theory [1, 2], should also be UV finite. That was indeed shown to be true to the two-loop orders and is expected to be true also to all orders [4].

It should be emphasized that we are interested in the reduced theory only as a tool for describing observables of the original string theory: it is the string theory that should dictate those quantities one should compute in the reduced theory[3].

Since the construction of the reduced theory from string theory equations of motion involves rewriting the theory in terms of the currents, the original superstring coordinates are effectively non-local functions of the new reduced theory variables. As was noticed in [1], the part of the reduced theory action given by the sum of the bosonic interaction potential and the fermionic 'Yukawa' term is essentially the same as the original $\mathrm{AdS}_5 \times S^5$ GS action expressed in terms of the new variables. This suggests that the two theories may actually be related by a non-trivial change of variables (from fields to currents) in the path integral, similar to the one used in the non-Abelian duality transformations (cf [22, 8]).

More precisely, the string theory path integral should contain delta functions of the Virasoro constraints, $\delta(T_{++})\delta(T_{--})$, and the change of variables from the supercoset coordinates to currents and to the reduced theory fields should solve these constraints. Heuristically, the additional gauged WZW and 'free' fermionic terms present in the reduced theory action may originate from the functional Jacobian of this change of variables.

[2] $\mathfrak{h}$ is the Lie algebra of the subgroup $H = SO(4) \times SO(4) = [SU(2)]^4$ of the group G.

[3] In particular, one may not be able to translate some characteristics of solitons in the reduced theory directly into meaningful quantities in string theory, etc. For example, the energies of the corresponding solutions in the reduced theory and in the string theory may be related (if at all) in a nontrivial way (cf [10]).

With this motivation in mind, here we propose the conjecture that the quantum string theory partition function (e.g., on a plane or on a cylinder) should be equal to the quantum reduced theory partition function,

$$\mathcal{Z}^{(\mathrm{II})}{}_{\textbf{string theory}} = \mathcal{Z}^{(\mathrm{II})}{}_{\textbf{reduced theory}}. \tag{1.1}$$

Since these two theories have the same number (8+8) of independent degrees of freedom this equality is obviously true in the trivial vacuum (BMN) case.

The aim of this paper is to provide evidence for this conjecture in the one-loop approximation, i.e. by expanding both sides of (1.1) near the corresponding classical solutions and computing the determinants of the quadratic fluctuation operators[4].

Given the classical equivalence between the string theory and the reduced theory the relation between the one-loop corrections which are determined by the quadratic fluctuation spectra may not look too surprising: after all, the quadratic fluctuation operators can be found from the classical equations of motion and thus should be expected to be in correspondence. However, given that the reduction procedure involves nontrivial steps of non-local change of variables and partial gauge fixing the general proof of the equivalence of the one-loop partition functions defined directly by the two actions appears to be non-trivial (and will not be attempted here).

Below we shall explicitly verify (1.1) in the one-loop approximation for a few simple classes of string solutions and their counterparts in the reduced theory: (i) generic string solutions localized in the $\mathrm{AdS}_2 \times S^2$ part of $\mathrm{AdS}_5 \times S^5$, and (ii) the homogeneous string solution representing a spinning string in S^3 part of S^5.

We shall start in section 2 with a review of the classical Pohlmeyer reduction for the $\mathrm{AdS}_5 \times S^5$ superstring theory following [1]. We shall mention the possibility of introducing an automorphism τ of the algebra of H in the construction of the reduced theory action (which then generalizes to an asymmetrically gauged G/H WZW model) and also comment on the vacuum structure of the reduced theory (section 2.2).

In section 3 we will consider the quadratic fluctuations of the conformal-gauge string theory equations of motion around classical string solutions. Here the fluctuating fields are string coordinates rather than currents but one can parametrize the dependence on the classical background in terms of the classical values of the current components. This allows one to start with a classical solution of the reduced theory and find the string fluctuation equations near the corresponding classical string solution. We shall apply this procedure to the case of generic $\mathrm{AdS}_2 \times S^2$ string solutions (section 3.2), preparing the ground for comparing with the fluctuation spectrum in the reduced theory.

In section 4 we will start with the action of the reduced theory and expand it to quadratic order near its classical solution. We will then specialize to the case of the reduced theory background corresponding to the generic string theory solution localized in the $\mathrm{AdS}_2 \times S^2$ subspace of $\mathrm{AdS}_5 \times S^5$ (section 4.2). Comparing to the quadratic fluctuation operators found on the string theory side in section 3 we will then be able to conclude that they match and thus (1.1) should be true at least in the one-loop approximation. The same conclusion will be reached in the case of homogeneous string solutions in $\mathbb{R}_t \times S^3$ and in $\mathrm{AdS}_3 \times S^1$ (section 4.3).

Section 5 will contain a summary and remarks on open problems.

[4] The classical parts of the partition functions determined by the values of the actions evaluated on the respective solutions will not match in general. The values of the two classical actions do not coincide on generic solutions which may not be surprising if part of the reduced theory action may be indeed interpreted as coming from the Jacobian of change of variables in the path integral. This is not a problem as the value of the reduced theory action on a classical solution is not necessarily an observable that one may be interested in on the string theory side.

In appendix A we will summarize some definitions and notation related to $PSU(2,2|4)$ supergroup and discuss decompositions of the corresponding superalgebra. In appendix B we shall relate the parametrization of the supercoset $\frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$ to standard embedding coordinates in $\mathrm{AdS}_5 \times S^5$. In appendix C we shall discuss some special cases of string solutions localized in the $\mathrm{AdS}_2 \times S^2$ part of $\mathrm{AdS}_5 \times S^5$ and the corresponding fluctuation equations in the reduced theory. Appendix D will contain a brief discussion of reduced theory counterparts of simple homogeneous string solutions. In appendix E we shall discuss an alternative way of computing the bosonic fluctuation frequencies in the reduced theory, using as an example the homogeneous solution discussed in section 4.3.1.

2. Review of the Pohlmeyer reduction of the $\mathrm{AdS}_5 \times S^5$ superstring

In this section we shall give a brief summary of the classical Pohlmeyer reduction for Type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ that follows [1].

We start with the 2D world-sheet sigma model arising from the Green–Schwarz action for the Type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ after fixing the conformal gauge. This is the F/G coset sigma model where $F = PSU(2,2|4)$ and $G = Sp(2,2) \times Sp(4)$ (see appendix A); we will henceforth call this sigma model the conformal-gauge string theory.

Let us consider the field $f \in PSU(2,2|4)$ and define the left-invariant current $J = f^{-1}\mathrm{d}f$. Under the $\mathbb{Z}_4$ decomposition discussed in appendix A the current can be written as follows:

$$J = f^{-1}\mathrm{d}f = \mathcal{A} + Q_1 + P + Q_2, \mathcal{A} \in \mathfrak{g}, \qquad Q_1 \in \mathfrak{f}_1,\ \ P \in \mathfrak{p},\ \ Q_2 \in \mathfrak{f}_3. \tag{2.1}$$

The GS action in the conformal gauge is then

$$L_{\mathrm{GS}} = \mathrm{STr}\big[P_+P_- + \tfrac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})\big], \tag{2.2}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$. We also need to impose the conformal-gauge (Virasoro) constraints,

$$\mathrm{STr}(P_\pm P_\pm) = 0. \tag{2.3}$$

This system has a G gauge symmetry under which,

$$\begin{aligned} &f \to fg \Rightarrow J \to g^{-1}Jg + g^{-1}dg,\\ &\qquad\quad \Rightarrow P \to g^{-1}Pg, \qquad \mathcal{A} \to g^{-1}\mathcal{A}g + g^{-1}dg,\\ &Q_1 \to g^{-1}Q_1g, \qquad Q_2 \to g^{-1}Q_2g. \end{aligned} \tag{2.4}$$

The equations of motion of the conformal-gauge string theory, obtained by varying f in (2.2), are

$$\begin{aligned} &\partial_+P_- + [\mathcal{A}_+, P_-] + [Q_{2+}, Q_{2-}] = 0,\\ &\partial_-P_+ + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] = 0,\\ &[P_+, Q_{1-}] = 0, [P_-, Q_{2+}] = 0. \end{aligned} \tag{2.5}$$

Interpreted as equations for the current components they should be supplemented by the Maurer–Cartan equation

$$\partial_-J_+ - \partial_+J_- + [J_-, J_+] = 0. \tag{2.6}$$

Under the $\mathbb{Z}_4$ decomposition the Maurer–Cartan equation (2.6) takes the form

$$\begin{aligned} &\partial_-P_+ - \partial_+P_- + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] + [P_-, \mathcal{A}_+] + [Q_{2-}, Q_{2+}] = 0,\\ &\partial_-\mathcal{A}_+ - \partial_+\mathcal{A}_- + [\mathcal{A}_-, \mathcal{A}_+] + [Q_{1-}, Q_{2+}] + [P_-, P_+] + [Q_{2-}, Q_{1+}] = 0,\\ &\partial_-Q_{1+} - \partial_+Q_{1-} + [\mathcal{A}_-, Q_{1+}] + [Q_{1-}, \mathcal{A}_+] + [P_-, Q_{2+}] + [Q_{2-}, P_+] = 0,\\ &\partial_-Q_{2+} - \partial_+Q_{2-} + [\mathcal{A}_-, Q_{2+}] + [Q_{1-}, P_+] + [P_-, Q_{1+}] + [Q_{2-}, \mathcal{A}_+] = 0. \end{aligned} \tag{2.7}$$

Here the first equation is automatically satisfied on the equations of motion (2.5).

The Pohlmeyer reduction procedure involves solving the equations of motion and the Virasoro constraints by introducing new variables parametrizing the physical degrees of freedom. The equations of motion of the reduced theory are then the final three equations in the decomposed Maurer–Cartan equation (2.7).

Let us briefly describe this reduction (for more details see section 6 of [1]). The polar decomposition theorem implies first that we can always use a G gauge transformation to set

$$P_+ = p_{1+}T_1 + p_{2+}T_2, \tag{2.8}$$

and second write P_- as follows

$$P_- = p_{1-}g^{-1}T_1 g + p_{2-}g^{-1}T_2 g, \tag{2.9}$$

where g is some element of $G = Sp(2,2)\times Sp(4)$ and $p_{1\pm}$ and $p_{2\pm}$ are functions of the world-sheet coordinates. T_1 and T_2 can be chosen as follows

$$\begin{aligned} T_1 &= \frac{\mathrm{i}}{2}\,\mathrm{diag}(1,1,-1,-1,0,0,0,0),\\ T_2 &= \frac{\mathrm{i}}{2}\,\mathrm{diag}(0,0,0,0,1,1,-1,-1). \end{aligned} \tag{2.10}$$

These two elements span the maximal Abelian subalgebra of $\mathfrak{p}$. To solve the Virasoro constraints we may then choose $p_+ = p_{1+} = p_{2+}$ and similarly, $p_- = p_{1-} = p_{2-}$. Thus

$$P_+ = p_+ T, \qquad P_- = p_- g^{-1}Tg, \tag{2.11}$$

where T is defined as follows

$$T = \frac{\mathrm{i}}{2}\,\mathrm{diag}(1,1,-1,-1,1,1,-1,-1). \tag{2.12}$$

T is an element of the maximal Abelian subalgebra of $\mathfrak{p}$. The group H is then defined as the subgroup of G which stabilizes T, that is $[h,\ T] = 0$, $h \in H$.

One way of fixing the κ-symmetry gauge is to project the fermionic currents onto the 'parallel space' (A.14) (see appendix A), i.e.

$$Q_1 = Q_1^{\parallel},\ gQ_2g^{-1} = (gQ_2g^{-1})^{\parallel}. \tag{2.13}$$

Substituting this into the fermionic equations of motion and noting that $[T, \mathfrak{f}_{1,3}^{\parallel}] = 2T\mathfrak{f}_{1,3}^{\parallel}$, it is possible to see that solving the resulting equations implies

$$Q_{1-} = Q_{2+} = 0. \tag{2.14}$$

The equations of motion (2.5) then become

$$\begin{aligned} &\partial_+ P_- + [\mathcal{A}_+, P_-] = 0,\\ &\partial_- P_+ + [\mathcal{A}_-, P_+] = 0. \end{aligned} \tag{2.15}$$

Using the residual conformal diffeomorphism symmetry it is always possible to set $p_\pm = \mu_\pm =$ const, so that we obtain

$$P_+ = \mu_+ T, \qquad P_- = \mu_- g^{-1}Tg. \tag{2.16}$$

It should be noted that if the sigma model were defined on 2D Minkowski space then we could use a Lorentz transformation to set $\mu_+ = \mu_- = \mu$ as was done in [1] (and originally assumed in [5]). However, if we are interested in the case of the closed string when the world-sheet is $\mathbb{R}\times S^1$ then this is not possible. It will be useful to define the following combination of μ_+ and μ_-,

$$\mu = \sqrt{\mu_+\mu_-}. \tag{2.17}$$

The equations of motion (2.15) can be solved as follows

$$\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1} + g^{-1}A_+ g, \qquad \mathcal{A}_- = A_-. \tag{2.18}$$

Here A_+ and A_- are arbitrary fields taking values in the algebra $\mathfrak{h}$ of H, i.e. $[A_\pm, T] = 0$. Finally, we make the following redefinitions of the non-vanishing fermionic fields

$$\Psi_R = \frac{1}{\sqrt{\mu_+}}(Q_{1+})^{\parallel}, \qquad \Psi_L = \frac{1}{\sqrt{\mu_-}}(gQ_{2-}g^{-1})^{\parallel}. \tag{2.19}$$

2.1. *Equations of motion and Lagrangian of reduced theory*

The equations of motion (2.5) and the Virasoro constraints (2.3) have been solved by writing the original currents in terms of a new set of fields, $(g\,, A_\pm,\ \Psi_R,\ \Psi_L)$, describing only the physical degrees of freedom of the system. Substituting these into the second, third and fourth equations in (2.7) we get the following set of equations of motion for the reduced theory

$$\begin{aligned} &\partial_-(g^{-1}\partial_+ g + g^{-1}A_+ g) - \partial_+ A_- + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g] \\ &\qquad = -\mu^2[g^{-1}Tg, T] - \mu[g^{-1}\Psi_L g, \Psi_R], \\ &D_-\Psi_R = \mu[T, g^{-1}\Psi_L g], \qquad D_+\Psi_L = \mu[T, g\Psi_R g^{-1}], \qquad D_\pm = \partial_\pm + [A_\pm,]. \end{aligned} \tag{2.20}$$

These equations naturally have $H \times H$ gauge symmetry,

$$\begin{aligned} &g \to h^{-1}g\bar{h}, \qquad A_+ \to h^{-1}A_+h + h^{-1}\partial_+ h, \qquad A_- \to \bar{h}^{-1}A_-\bar{h} + \bar{h}^{-1}\partial_-\bar{h} \\ &\Psi_R \to \bar{h}^{-1}\Psi_R\bar{h}, \qquad \Psi_L \to h^{-1}\Psi_L h. \end{aligned} \tag{2.21}$$

The factor of H that corresponds to acting from the right on g arises as a subgroup from the original G gauge freedom in the conformal-gauge string theory. The reason is that once P_+ has been rotated to be proportional to T, it is still possible to perform further G gauge transformations retaining this structure, as long as $g \in H$. The other factor of H, which corresponds to acting from the left on g arises because in defining the reduced theory field g, there is an ambiguity: it is possible to let $g \to hg$, where h is an arbitrary element of H, without changing that P_- is proportional to $g^{-1}Tg$. Both of these gauge freedoms come about because H is the stabilizer of T (i.e., $[h,\ T] = 0$ for $h \in H$).

To be able to write a sensible Lagrangian which leads to the equations of motion (2.20) we need to partially fix the $H \times H$ gauge symmetry to a H gauge symmetry. We can do this by demanding that

$$\begin{aligned} \tau(A_+) &= \left(g^{-1}\partial_+ g + g^{-1}A_+ g - \tfrac{1}{2}[[T, \Psi_R], \Psi_R]\right)_{\mathfrak{h}}, \\ \tau^{-1}(A_-) &= \left(-\partial_- g g^{-1} + gA_- g^{-1} - \tfrac{1}{2}[[T, \Psi_L], \Psi_L]\right)_{\mathfrak{h}}. \end{aligned} \tag{2.22}$$

Here τ (not to be confused with a time-like world-sheet coordinate) is a supertrace-preserving[5] automorphism of the algebra $\mathfrak{h}$. As discussed in [1], this partial gauge fixing is always possible[6]. The gauge symmetry is now reduced to the following asymmetric H gauge symmetry,

$$\begin{aligned} &g \to h^{-1}g\hat{\tau}(h), \qquad A_+ \to h^{-1}A_+h + h^{-1}\partial_+ h, \qquad A_- \to \hat{\tau}(h)^{-1}A_-\hat{\tau}(h) + \hat{\tau}(h)^{-1}\partial_-\hat{\tau}(h) \\ &\qquad \Psi_R \to \hat{\tau}(h)^{-1}\Psi_R\hat{\tau}(h), \qquad \Psi_L \to h^{-1}\Psi_L h, \end{aligned} \tag{2.23}$$

where $\hat{\tau}$ is a lift of τ from $\mathfrak{h}$ to H.

The equations of motion (2.20) and the gauge constraints (2.22) then follow from the following Lagrangian[7],

$$\begin{aligned} L_{\text{tot}} = L_{\text{gWZW}} &+ \mu^2\,\text{STr}(g^{-1}TgT) + \tfrac{1}{2}\text{STr}(\Psi_L[T, D_+\Psi_L] \\ &+ \Psi_R[T, D_-\Psi_R]) + \mu\,\text{STr}(g^{-1}\Psi_L g\Psi_R), \end{aligned} \tag{2.24}$$

[5] $\text{STr}(\tau(u_1)\tau(u_2)) = \text{STr}(u_1u_2)$, $u_{1,2} \in \mathfrak{h}$.

[6] Compared to [1], we choose to redefine $A_- \to \tau^{-1}(A_-)$.

[7] The overall coefficient in the reduced theory action should be the same string tension that appears in the $AdS_5 \times S^5$ string action.

where $L_{\rm gWZW}$ is the Lagrangian of the asymmetrically gauged G/H WZW model,

$$I_{\rm gWZW} = \int \frac{{\rm d}^2\sigma}{4\pi} {\rm STr}(g^{-1}\partial_+ g g^{-1}\partial_- g) - \int \frac{{\rm d}^3\sigma}{12\pi} {\rm STr}(g^{-1}{\rm d}g g^{-1}{\rm d}g g^{-1}{\rm d}g)$$
$$+ \int \frac{{\rm d}^2\sigma}{2\pi} {\rm STr}(A_+\,\partial_- g g^{-1} - A_-\, g^{-1}\partial_+ g - g^{-1}A_+ g A_- + \tau(A_+)A_-)\,. \qquad (2.25)$$

This Lagrangian is invariant under the gauge transformations (2.23) as expected.

The reduced theory is thus the G/H asymmetrically gauged WZW model with a gauge-invariant integrable potential and fermionic extension. For the case of the superstring on $AdS_5 \times S^5$ we have $G = Sp(2,2) \times Sp(4)$ and $H = [SU(2)]^4$. The embedding of these subgroups into $PSU(2,2|4)$ that we use is discussed in appendix A.

Let us stress that the equations of motion (2.20) obtained directly from string theory equations after solving the Virasoro conditions in terms of new current variables do not 'know' about the τ-automorphism. Thus the information contained in (2.24) with (2.25) that is relevant for string theory should also not depend on τ. However, it is not clear *a priori* (and seems unlikely) that the reduced theory actions with different choices of τ are completely equivalent as 2D quantum field theories.

In the sections 3 and 4 we shall consider the case of the symmetric gauge fixing when the automorphism τ is trivial, i.e. the reduced theory Lagrangian is given by (2.24), (2.25) with $\tau = \mathbf{1}$.

2.2. *Vacua of the reduced theory*

The vacua of the reduced theory may be defined as constant solutions which minimize the potential $-\mu^2\,{\rm STr}(g^{-1}TgT)$ in (2.24). These are then

$$g_{\rm vac} = h_0 \in H, \qquad h_0 = {\rm const.} \qquad (2.26)$$

Back in string theory all these vacua are equivalent to the BMN vacuum. As discussed above, when carrying out the reduction we initially have the equations (2.20) with $H \times H$ gauge symmetry, (2.21). We then use some of this gauge symmetry to fix the gauge fields as in (2.22).

Before this partial gauge fixing it is always possible to choose the vacuum in the equations (2.20) to be the identity, $g = \mathbf{1}$: choices of $g_{\rm vac} = h_0 \in H$ are gauge equivalent. After the gauge fixing needed to get a Lagrangian set of equations of motion this is no longer so: we get a space of vacua (2.26) that are not related by the residual H gauge transformations. Still, they should be effectively equivalent as far as the information relevant for string theory is concerned.

Let us emphasize that ultimately we are interested in observables of the string theory. We are only interested in observables of the reduced theory in the sense of what they say about the observables in the string theory. At the level of the equations of motion (i.e., classically) it is clear that the latter should not depend on a particular $H \times H \to H$ gauge fixing. As the one-loop corrections are essentially determined by the equations of motion, this should also be true at the one-loop level (and should hopefully be true in general).

It is useful to note that expanding the reduced theory action near different vacua is related to using different partial gauge-fixings or different choices of τ in (2.22). Indeed, it is easy to see that starting with the action (2.25) with $\tau = \mathbf{1}$ and expanding it near $g = h_0$ is equivalent to starting with (2.25) with the special choice of the automorphism $\tau(u) = h_0^{-1}uh_0$ and expanding it near $g = \mathbf{1}$.

As was mentioned in [1], there is an apparent problem with expanding the symmetrically gauged ($\tau = \mathbf{1}$) action (2.24) near the trivial vacuum, $g_{\rm vac} = \mathbf{1}$: the $A_+A_- - g^{-1}A_+gA_-$ part of the action (2.25) is then degenerate. This complication may be by-passed by exploiting the freedom to choose a different gauging or a different vacuum in (2.26) to expand around. For example, one may expand the symmetrically gauged model near

$$g_{\mathrm{vac}} = \begin{pmatrix} \mathrm{i}\sigma_i & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathrm{i}\sigma_j & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathrm{i}\sigma_k & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathrm{i}\sigma_l \end{pmatrix}, \tag{2.27}$$

which is a constant matrix in $H = [SU(2)]^4$. Here $\sigma_{1,2,3}$ are the Pauli matrices and i, j, k, l can take any values $1, 2, 3$. It should be noted that these choices are all related to each other by symmetric H gauge transformations, but are not equivalent to $g_{\mathrm{vac}} = \mathbf{1}$. Expanding near this vacuum (combined with an appropriate H gauge fixing) then removes the degeneracy. This observation may be useful for a future study of the S-matrix of the reduced theory.

If we start with the symmetrically gauged WZW model we may parametrize g in terms of eight bosonic scalar fields (after H gauge fixing).[8] We should do this so that when these fields all vanish we are left with $g_{\mathrm{vac}} = h_0$, for constant $h_0 \in H$ (this includes $g_{\mathrm{vac}} = \mathbf{1}$ and also g_{vac} as given in (2.27)). At the level of the equations of motion (2.20) these choices are all related by $H \times H$ gauge transformations, but not by symmetric H gauge transformations. Therefore, there will be many solutions of the symmetrically gauged WZW model, which are not related by H gauge transformations, but which correspond to the same classical string solution (as they are related by a $H \times H$ gauge transformation, ignoring the gauge constraints). They may be distinguished by the vacuum they approach in the limit when the string solution shrinks to a point.

In most of this paper we will always look for classical solutions of the reduced theory such that they are solutions of the symmetrically gauged WZW model and have a vacuum limit that is related by a H gauge transformation to (2.27).[9]

3. Fluctuations near classical solution from string theory equations of motion

In this section we shall discuss fluctuations of the conformal-gauge string theory around classical string solutions at the level of the equations of motion. The underlying motivation is to compare one-loop quantum corrections in string theory and the reduced theory. Since the classical equations of the reduced theory are closely related to the original conformal-gauge string equations (and their classical solutions are in direct correspondence) the fluctuation spectra near the respective solutions should also be closely related.

As discussed above, the string theory equations can be written in terms of the current components built out of the field $f \in PSU(2, 2|4)$. Rather than fluctuating the currents directly here we will first fluctuate f and then consider how this affects the equations of motion and the Maurer–Cartan equations for the currents.

It is possible to parametrize f in terms of fields that can be viewed as coordinates on $\mathrm{AdS}_5 \times S^5$. The parametrization that we use is discussed in appendix B. Thus fluctuating f is equivalent to fluctuating these embedding coordinates. It is still advantageous to write the

[8] Below we will not explicitly relate g to string coordinates (we will always embed the string coordinates into f and compute g following the procedure outlined in section 2).

[9] In appendix D we will consider the complex sine-Gordon and complex sinh-Gordon models as truncated reduced theory models corresponding to the bosonic part of superstring theory on $\mathrm{AdS}_3 \times S^3$. When considering these models we have already implicitly chosen a particular parametrization of g in terms of scalar fields, or, equivalently, a particular embedding of the string coordinates in g. This parametrization is different from the one used in the rest of the paper.

classical equations of motion in terms of the currents as then the resulting fluctuation equations retain the algebra structure.

One may use the Pohlmeyer reduction to simplify the fluctuations of the conformal-gauge string theory. Starting with a classical solution of the reduced theory, if we are interested in the fluctuation spectrum we do not need to reconstruct the corresponding classical form of f: we need only to know the corresponding classical string theory currents. We can then substitute the reconstructed currents into the fluctuation equations of the conformal-gauge string theory.

This simplifies the fluctuation equations because in the Pohlmeyer reduction the G gauge freedom of the conformal-gauge string theory is used to rotate P_+ such that it is proportional to T (see section 2). In terms of the embedding coordinates on $AdS_5 \times S^5$, this is equivalent to choosing the coordinate system such that one of the directions of the world-sheet always lies in a particular direction. The massless fluctuations, which are removed via the Virasoro constraints, are the two fluctuations in the directions along the world-sheet, while the physical fluctuations are those transverse to the world-sheet. Since the Virasoro constraints are already solved in the reduced theory it turns out to be much easier to isolate the physical fluctuations.

Below we shall study in detail a general class of classical solutions living in an $AdS_2 \times S^2$ subspace of $AdS_5 \times S^5$ and consider the functional determinants of the operators acting on the physical fluctuations[10]. For some special solutions we will see that the results will agree with the previously found ones, such as for fluctuations near the giant magnon solution [33].

The Lagrangian and the equations of motion for the conformal-gauge string theory are given in (2.2) and (2.5), respectively. We start with a classical solution f_0 (with the corresponding current $J_0 = \mathcal{A}_0 + Q_{1\,0} + P_0 + Q_{2\,0}$), and set

$$f = f_0 e^{\xi}, \qquad \xi \in \mathfrak{psu}(2,2|4). \tag{3.1}$$

This should then be substituted into the classical equations of motion (2.5) and the Virasoro constraints (2.3). The resulting equations are then expanded to first order in the fluctuation field ξ. Since

$$\begin{aligned} J = f^{-1}\,\mathrm{d}f &= f_0^{-1}\,\mathrm{d}f_0 + \left[f_0^{-1}\,\mathrm{d}f_0, \xi\right] + \mathrm{d}\xi + \mathcal{O}(\xi^2) + \\ &= J_0 + [J_0, \xi] + \mathrm{d}\xi + \mathcal{O}(\xi^2) \end{aligned} \tag{3.2}$$

is flat, $\mathrm{d}J + J \wedge J = 0$, the fluctuation equations arising from the Maurer–Cartan equations will be satisfied automatically.

We can split the fluctuation field ξ under the $\mathbb{Z}_4$ decomposition

$$\xi = \xi_0 + \xi_1 + \xi_2 + \xi_3. \tag{3.3}$$

Under the $\mathbb{Z}_4$ grading J decomposes as follows to first order in ξ,

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_0 + [\mathcal{A}_0, \xi_0] + [Q_{10}, \xi_3] + [P_0, \xi_2] + [Q_{20}, \xi_1] + \mathrm{d}\xi_0 \\ &= \mathcal{A}_0 + \delta\mathcal{A}, \\ P &= P_0 + [\mathcal{A}_0, \xi_2] + [Q_{10}, \xi_1] + [P_0, \xi_0] + [Q_{20}, \xi_3] + \mathrm{d}\xi_2 \\ &= P_0 + \delta P, \\ Q_1 &= Q_{10} + [\mathcal{A}_0, \xi_1] + [Q_{10}, \xi_0] + [P_0, \xi_3] + [Q_{20}, \xi_2] + \mathrm{d}\xi_1 \\ &= Q_{10} + \delta Q_1, \\ Q_2 &= Q_{20} + [\mathcal{A}_0, \xi_3] + [Q_{1,0}, \xi_2] + [P_0, \xi_1] + [Q_{20}, \xi_0] + \mathrm{d}\xi_3 \\ &= Q_{20} + \delta Q_2. \end{aligned} \tag{3.4}$$

[10] Note that here while we only consider classical solutions living in $AdS_2 \times S^2$ we fluctuate the canonical field f in all directions, including the fermionic directions.

Substituting these relations back into (2.5) gives the equations of motion for the fluctuations ξ. These need to be supplemented by the equations which arise from substituting (3.4) back into the Virasoro conditions (2.3) which will give the constraint equations which remove the two massless 'longitudinal' bosonic fluctuations.

3.1. Fluctuations of string equations around a classical solution of the reduced theory

The aim of this section is to determine the fluctuations around a classical solution of the conformal-gauge string theory corresponding to a solution of the reduced theory. Again, the eventual goal is to show the equivalence between the fluctuation spectrum and thus the one-loop corrections in string theory and in the reduced theory in (cf section 4).

The strategy is to start with a classical solution of the reduced theory and then reconstruct the classical currents of the conformal-gauge string theory. As already mentioned, we do not need to reconstruct the full classical solution of the conformal-gauge string theory, f_0. Given a classical solution of the reduced theory g_0, $A_{\pm 0}$, Ψ_{R0}, Ψ_{L0}, the reconstructed currents of the string theory solution are as follows

$$\begin{aligned} P_{0+} &= \mu_+ T, & P_{0-} &= \mu_- g_0^{-1} T g_0, \\ \mathcal{A}_{0+} &= g_0^{-1}\partial_+ g_0 + g^{-1} A_{+0} g_0, & \mathcal{A}_{0-} &= A_{-0}, \\ Q_{10+} &= \sqrt{\mu_+}\Psi_{R0}, & Q_{10-} &= 0, \\ Q_{20+} &= 0, & Q_{20-} &= \sqrt{\mu_-} g_0^{-1}\Psi_{L0} g_0. \end{aligned} \tag{3.5}$$

Motivated by the comparison to the reduced theory let us make the following redefinitions of the fermionic components of ξ

$$\hat{\xi}_1 = \frac{\xi_1}{\sqrt{\mu_-}}, \qquad \hat{\xi}_3 = -\frac{g_0 \xi_3 g_0^{-1}}{\sqrt{\mu_+}}. \tag{3.6}$$

We then fix the κ-symmetry gauge by choosing $\hat{\xi}_1 = \hat{\xi}_1^{\parallel}$ and $\hat{\xi}_3 = \hat{\xi}_3^{\parallel}$.

Substituting these formulae into (2.5) and (2.3) gives the equations of motion and constraint equations for the fluctuations. Here we will give these equations for the fluctuations with vanishing classical fermionic content, i.e. $Q_{10} = Q_{20} = \Psi_{R0} = \Psi_{L0} = 0$. We will also assume that the classical solution of the reduced theory has vanishing gauge fields, that is $A_{\pm 0} = 0$. It is possible to see that using the H gauge freedom and the fact that the current A_0 is flat,[11] it is always possible to choose the classical solution of the reduced theory equations (2.20) such that $A_{\pm 0} = 0$. The fluctuation equations are then

$$\begin{aligned} &\partial_+\partial_-\xi_2 + \partial_-\left[g_0^{-1}\partial_+ g_0, \xi_2\right] + \mu^2\left[\left[g_0^{-1} T g_0, \xi_2\right], T\right] = 0, \\ &\partial_-\hat{\xi}_1 + \mu[T, [T, g_0^{-1} T, \hat{\xi}_3] g_0]] = 0, \\ &\partial_+\hat{\xi}_3 + \mu[T, [T, g_0^{-1}[T, \hat{\xi}_3 g_0]] = 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\mathrm{STr}\left(T\left(\left[g_0^{-1}\partial_+ g_0, \xi_2\right] + \partial_+\xi_2\right)\right) = 0, \\ &\mathrm{STr}\left(g_0^{-1} T g_0 \partial_-\xi_2\right) = 0. \end{aligned} \tag{3.8}$$

Here we have used that $T^2 = -\frac{1}{4}\mathbf{1}_8$ and the cyclicity of the supertrace to simplify the fluctuations of the Virasoro constraints.

[11] The flatness of A_0 comes from the equations of motion and thus implies that this is a statement that can only be made on-shell.

3.2. Special case: solutions in $AdS_2 \times S^2$ subspace of $AdS_5 \times S^5$

Let us consider a particular case of the classical string solutions in a $\mathrm{AdS}_2 \times S^2$ subspace of $\mathrm{AdS}_5 \times S^5$ and the corresponding classical solutions in the reduced theory (see also appendix B.1 for details). We will see that the resulting functional determinants, which determine the one-loop corrections, match the corresponding functional determinants in the reduced theory computed in section 4.2.

There are many interesting string solutions which live in $\mathrm{AdS}_2 \times S^2$, and using the results of this section it may be possible to better understand the one-loop corrections to their energies. The simplest are the ones that effectively live in $\mathbb{R}_t \times S^1$, that is the point-like orbiting string (i.e., the geodesic corresponding to the BMN vacuum state) and the (unstable) static string wrapped on a big circle of S^5. There are no homogeneous string solutions in $\mathrm{AdS}_2 \times S^2$ apart from these two special cases, but there are many other simple configurations: pulsating strings, folded strings and finite-size magnons (see [34, 35, 14, 12, 13] and references therein). One of the limits of the finite-size magnon is the giant magnon [10], for which the one-loop correction was shown to vanish [33]. We shall compare our results against the expressions in this paper in appendix C and show that they agree.

As discussed in appendix B.1, for the bosonic solutions in $\mathrm{AdS}_2 \times S^2$ we can consider the following element of G as the field used to parametrize P_- in the reduction procedure (2.11)

$$g_0 = \begin{pmatrix} g_A & \mathbf{0}_4 \\ \mathbf{0}_4 & g_S \end{pmatrix}, \tag{3.9}$$

$$g_A = \begin{pmatrix} \mathrm{i}\cosh\phi_A & 0 & 0 & \sinh\phi_A \\ 0 & -\mathrm{i}\cosh\phi_A & \sinh\phi_A & 0 \\ 0 & \sinh\phi_A & \mathrm{i}\cosh\phi_A & 0 \\ \sinh\phi_A & 0 & 0 & -\mathrm{i}\cosh\phi_A \end{pmatrix},$$

$$g_S = \begin{pmatrix} \mathrm{i}\cos\phi_S & 0 & 0 & \mathrm{i}\sin\phi_S \\ 0 & -\mathrm{i}\cos\phi_S & \mathrm{i}\sin\phi_S & 0 \\ 0 & \mathrm{i}\sin\phi_S & \mathrm{i}\cos\phi_S & 0 \\ \mathrm{i}\sin\phi_S & 0 & 0 & -\mathrm{i}\cos\phi_S \end{pmatrix},$$

where ϕ_A and ϕ_S satisfy

$$\partial_+\partial_-\phi_A + \frac{\mu^2}{2}\sinh 2\phi_A = 0, \qquad \partial_+\partial_-\phi_S + \frac{\mu^2}{2}\sin 2\phi_S = 0. \tag{3.10}$$

For this configuration $A_{\pm 0} = \Psi_{R0} = \Psi_{L0} = 0$ (see appendix B.1). It should be noted that as $g_0^{-1}\partial_+ g_0 \in \mathfrak{m}$ and $\partial_- g_0 g_0^{-1} \in \mathfrak{m}$ (as defined in appendix A), $A_{\pm 0} = 0$ is a consistent solution for the gauge fields. This configuration satisfies the classical equations of motion of the reduced theory, provided ϕ_A and ϕ_S satisfy the above sinh-Gordon and sine-Gordon equations.

3.2.1. Fluctuations near $AdS_2 \times S^2$ solution in string theory. The fluctuations around the corresponding solution in the conformal-gauge string theory can be found following the method in section 3.1. That is, we start from the reduced theory classical solution, reconstruct the classical currents of the string theory, and substitute these into the equations of motion for the fluctuations of the field f.

Substituting (3.9) into (3.7) and considering the components of the resulting matrix equations the following fluctuation equations arise. In the bosonic AdS_5 sector we obtain

$$\partial_+\partial_- z_i + \mu^2\cosh 2\phi_A\, z_i \equiv \mathcal{O}_1 \quad z_i = 0, \qquad i = 1, 2, 3 \tag{3.11}$$

and one copy of the following set of coupled equations

$$\partial_+\partial_- z_4 + \mu^2 \cosh 2\phi_A \, z_4 - 2\partial_+\phi_A \partial_- z_5 = 0, \tag{3.12}$$

$$\partial_-(\partial_+ z_5 - 2\partial_+\phi_A z_4) = 0. \tag{3.13}$$

Here $z_1,\ z_2,\ z_3,\ z_4,\ z_5$ are the five components of ξ_2 in the AdS_5 sector.

We still need to impose the constraints on the fluctuations arising from fluctuating the Virasoro constraints (3.8). For the present configuration the AdS_5 and S^5 sectors are decoupled and thus in the Virasoro constraints we can split up the supertrace into traces over the two sectors and demand that they both vanish separately. The following constraints then arise for the fluctuations in the AdS_5 sector,

$$\partial_+ z_5 - 2\partial_+\phi_A z_4 = 0, \qquad \partial_- z_5 - \tanh 2\phi_A \partial_- z_4 = 0. \tag{3.14}$$

It is possible to see that both (3.12) and (3.13) are implied by (3.14). This coupled first-order system is equivalent to the second-order system, obtained by eliminating z_4 or z_5. Thus the relevant fluctuation operator can be found by either eliminating z_4 or z_5 from (3.14) or by just considering the coupled first-order operator. These should lead to the same functional determinant.

Here we choose to eliminate z_5, resulting in the following equation for the fourth physical bosonic fluctuation z_4 in the AdS_5 sector (the other three are given by (3.11))

$$\partial_+\partial_- z_4 + \mu^2 \cosh 2\phi_A \, z_4 - 2\tanh 2\phi_A \partial_+\phi_A \partial_- z_4 \equiv \mathcal{O}_2 \, z_4 = 0. \tag{3.15}$$

Let us show that the determinants of the two operators,

$$\mathcal{O}_1 = \partial_+\partial_- + \mu^2 \cosh 2\phi_A \tag{3.16}$$

and

$$\mathcal{O}_2 = \partial_+\partial_- + \mu^2 \cosh 2\phi_A - 2\tanh 2\phi_A \partial_+\phi_A \partial_- \tag{3.17}$$

are equal. Defining

$$V_A \equiv \mu^2 \cosh 2\phi_A, \tag{3.18}$$

we have

$$\mathcal{O}_1 = \partial_+\partial_- + V_A \qquad \text{and} \qquad \mathcal{O}_2 = (V_A\partial_+)\left(V_A^{-1}\partial_-\right) + V_A. \tag{3.19}$$

Considering the product

$$\begin{pmatrix} V_A & 0 \\ 0 & V_A^{-1} \end{pmatrix} \begin{pmatrix} \partial_+ & -1 \\ V_A & \partial_- \end{pmatrix} = \begin{pmatrix} V_A\partial_+ & -V_A \\ 1 & V_A^{-1}\partial_- \end{pmatrix} \tag{3.20}$$

and taking the determinant[12] on both sides we immediately see that

$$\det \mathcal{O}_1 = \det \mathcal{O}_2. \tag{3.21}$$

Therefore, the contribution of the AdS_5 sector to the one-loop correction is given by the four copies of the determinant of $\mathcal{O}_1$, i.e.

$$4 \ln \det(\partial_+\partial_- + \mu^2 \cosh 2\phi_A). \tag{3.22}$$

In the bosonic S^5 sector the story is the same, with $V_A \to V_S = \mu^2 \cos 2\phi_S$. The contribution of this sector is then given by

$$4 \ln \det(\partial_+\partial_- + \mu^2 \cos 2\phi_S). \tag{3.23}$$

[12] For a matrix of operators we have $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - ACA^{-1}B) = \det(DA - CA^{-1}BA)$.

For the fermionic fluctuations we get the following sets of coupled equations

$$\begin{aligned}
&\partial_-\vartheta_i + \mu\cos\phi_S\cosh\phi_A\vartheta'_i + \mu\sin\phi_S\sinh\phi_A\vartheta'_{i+1} = 0,\\
&\partial_+\vartheta'_i - \mu\cos\phi_S\cosh\phi_A\vartheta_i + \mu\sin\phi_S\sinh\phi_A\vartheta_{i+1} = 0, \qquad i = 1, 3, 5, 7\\
&\partial_-\vartheta_{i+1} + \mu\cos\phi_S\cosh\phi_A\vartheta'_{i+1} - \mu\sin\phi_S\sinh\phi_A\vartheta'_i = 0,\\
&\partial_+\vartheta'_{i+1} - \mu\cos\phi_S\cosh\phi_A\vartheta_{i+1} - \mu\sin\phi_S\sinh\phi_A\vartheta_i = 0.
\end{aligned} \tag{3.24}$$

Here the anticommuting functions ϑ_k are components of $\hat{\xi}_1 \in \mathfrak{f}_1^{\parallel}$ and ϑ'_k are components of $\hat{\xi}_3 \in \mathfrak{f}_3^{\parallel}$. The 16 coupled first-order equations can be rearranged into eight coupled second-order equations describing the expected eight fermionic degrees of freedom.

In appendix C the results of this section are applied to the case of the giant magnon classical solution [10] and shown to agree with [33], where the one-loop correction to the energy was computed by fluctuating the embedding coordinates.

3.2.2. Solutions in $\mathbb{R}_t \times S^1$. There are two special string solutions that live in $\mathbb{R}_t \times S^1$ with $\mathbb{R}_t$ from AdS_5 and S^1 from S^5. These are (i) the (supersymmetric) point-like orbiting string, $t = \kappa\tau,\ \theta = \kappa\tau$, and (ii) the (unstable) static wound closed string, $t = k\tau,\ \theta = k\sigma,\ k \in \mathbb{Z}$ (k is the winding number). Here t and θ are the coordinates in AdS_5 and S^5 as defined in appendix B.

The reduced theory solutions corresponding to these two string solutions are the constant solutions of the sinh-Gordon and sine-Gordon equations. For the sinh-Gordon one the only constant solution is $\phi_A = 0$. For the sine-Gordon equation the constant solutions are $\phi_S = \frac{n\pi}{2}$, $n \in \mathbb{Z}$. These break down into two distinct types, either $\phi_S = n\pi$ or $\phi_S = n\pi + \frac{\pi}{2}$, which correspond to minima and maxima of the potential, $\mu^2\cos 2\phi_S$; these lead to stable and unstable solutions, respectively.

The reduced theory solution $\phi_A = \phi_S = 0$ gives the point-like string in $\mathbb{R}_t \times S^1$ in string theory, with $\mu = \kappa$. Thus a stable vacuum solution of the reduced theory corresponds to the stable BMN vacuum solution of the conformal-gauge string theory. The bosonic and fermionic fluctuation equations are then the familiar one

$$\partial_+\partial_-\zeta_i + \mu^2\zeta_i = 0, \qquad i = 1, \ldots, 8. \tag{3.25}$$

$$\partial_+\partial_-\vartheta_i + \mu^2\vartheta_i = 0, \qquad i = 1, \ldots, 8. \tag{3.26}$$

For the static string wrapped on S^1 in S^5 the corresponding reduced theory solution is $\phi_A = 0,\ \phi_S = \frac{\pi}{2}$, with $\mu = k$. As expected, an unstable solution in the reduced theory gives rise to an unstable solution in string theory. The bosonic AdS_5 and S^5 fluctuation equations are respectively

$$\partial_+\partial_-\zeta_i + \mu^2\zeta_i = 0, \qquad i = 1, \ldots, 4. \tag{3.27}$$

$$\partial_+\partial_-\zeta_i - \mu^2\zeta_i = 0, \qquad i = 5, \ldots, 8. \tag{3.28}$$

The fermionic fluctuation equations are

$$\partial_+\partial_-\vartheta_i = 0, \qquad i = 1, \ldots, 8. \tag{3.29}$$

For both the above solutions the fluctuation spectra computed in the reduced theory and directly in the string theory (using, e.g., the embedding coordinates) match, and thus the one-loop partition functions also match, providing a simple check of our general claim.

4. Fluctuations near a classical solution from the action of the reduced theory

In this section we will investigate the quadratic fluctuations in the reduced theory action expanded around classical solutions. Again, the aim is to see whether the sum of logarithms of the functional determinants which gives the one-loop partition function of the reduced theory is the same as in the conformal-gauge string theory expanded near the corresponding solution.

While we will not prove in general that the one-loop partition functions match, we shall demonstrate the equivalence for certain classes of classical solutions. These include solutions which live in an $\mathrm{AdS}_2 \times S^2$ subspace of $\mathrm{AdS}_5 \times S^5$ and also homogeneous solutions of the conformal-gauge string theory.

We shall parametrize the basic variable of the reduced theory $g \in G$ as follows

$$g = g_0 e^{\eta}, \qquad \eta \in \mathfrak{g}, \tag{4.1}$$

where η is the fluctuation field.

Under the $\mathbb{Z}_2$ decomposition discussed in appendix A we have $\eta = \eta^{\parallel} + \eta^{\perp}$ where $\eta^{\parallel} \in \mathfrak{m}$ and $\eta^{\perp} \in \mathfrak{h}$. As the physical bosonic fluctuations should be those corresponding to the coset G/H part, we will take them to be the components of $\eta^{\parallel}$.[13] As expected, there are eight independent components of the bosonic fluctuation field $\eta^{\parallel}$.

The fields $\eta^{\perp}$ and the fluctuations of the gauge fields, $\delta A_{\pm} \in \mathfrak{h}$, will, in general, be coupled to $\eta^{\parallel}$. To isolate the physical fluctuations the H gauge needs to be fixed. We will always choose to fix the gauge on $\eta^{\perp}$ and $\delta A_{\pm}$, understanding that the components of $\eta^{\parallel}$ should be the physical fluctuations.

An evidence that the components of $\eta^{\parallel}$ are the physical fluctuations is that in the quadratic fluctuation Lagrangian (4.3), the kinetic term is given by

$$\mathrm{STr}(\partial_+\eta\partial_-\eta) = \mathrm{STr}(\partial_+\eta^{\parallel}\partial_-\eta^{\parallel}) + \mathrm{STr}(\partial_+\eta^{\perp}\partial_-\eta^{\perp}).$$

Expressing η in terms of the component fields gives kinetic terms with the correct sign for the fields in $\eta^{\parallel}$, but the wrong sign for the some of the fields in $\eta^{\perp}$.

It should be noted that under the H gauge transformations we have

$$\eta \to h^{-1}\eta h \quad \Rightarrow \quad \eta^{\parallel} \to h^{-1}\eta^{\parallel} h, \qquad \eta^{\perp} \to h^{-1}\eta^{\perp} h.$$

Therefore, the components of $\eta^{\parallel}$ and $\eta^{\perp}$ cannot mix under these transformations.

4.1. Expansion of the reduced theory action

The reduced theory action found in [1, 2] is a particular fermionic extension of the G/H left–right symmetrically gauged WZW model with a H gauge invariant integrable potential (for its detailed discussion see also [4]). In the case of the $\mathrm{AdS}_5 \times S^5$ superstring we have $G = Sp(2,2) \times Sp(4)$ and $H = [SU(2)]^4$. The embedding of these subgroups into $PSU(2,2|4)$ that we use is discussed in appendix A. The Lagrangian and the equations of motion for this theory were given in (2.24) and (2.20), respectively.

We consider the fluctuations around a classical solution, g_0, $A_{\pm 0}$, Ψ_{R0}, Ψ_{L0}, as follows

$$\begin{aligned} &g = g_0 e^{\eta} = g_0\big(1 + \eta + \tfrac{1}{2}\eta^2 + \mathcal{O}(\eta^3)\big),\\ &A_+ = A_{+0} + \delta A_+, \qquad A_- = A_{-0} + \delta A_-,\\ &\Psi_R = \Psi_{R0} + \delta\Psi_R, \qquad \Psi_L = \Psi_{L0} + \delta\Psi_L. \end{aligned} \tag{4.2}$$

[13] One may think of these fluctuations as corresponding to Cartesian coordinates (as opposed to radii and angles), cf (4.7).

Below we will only consider classical solutions with a vanishing fermionic content, i.e. Ψ_{R0} and Ψ_{L0} will be set to zero. The quadratic fluctuation part of the Lagrangian (2.24) is then

$$\begin{aligned} L_{\text{quad}} = \mathrm{STr}\big[&\tfrac{1}{2}\partial_+\eta\partial_-\eta + \tfrac{1}{2}(\eta\partial_-\eta - \partial_-\eta\eta)g_0^{-1}\partial_+g_0 + \delta A_+g_0\partial_-\eta g_0^{-1} - \tfrac{1}{2}A_{+0}g_0\partial_-\eta\eta g_0^{-1} \\ &+ \tfrac{1}{2}A_{+0}g_0\eta\partial_-\eta g_0^{-1} + \delta A_-\eta g_0^{-1}\partial_+g_0 - \delta A_-g_0^{-1}\partial_+g_0\eta - \delta A_-\partial_+\eta + A_{-0}\eta g_0^{-1}\partial_+g_0\eta \\ &+ \tfrac{1}{2}A_{-0}\eta\partial_+\eta - \tfrac{1}{2}A_{-0}\eta^2g_0^{-1}\partial_+g_0 - \tfrac{1}{2}A_{-0}g_0^{-1}\partial_+g_0\eta^2 - \tfrac{1}{2}A_{-0}\partial_+\eta\eta + \delta A_+\delta A_- \\ &- \tfrac{1}{2}\eta^2g_0^{-1}A_{+0}gA_{-0} - \tfrac{1}{2}g_0^{-1}A_{+0}g_0\eta^2A_{-0} + \eta g_0^{-1}\delta A_+g_0A_{-0} + \eta g_0^{-1}A_{+0}g_0\eta A_{-0} \\ &+ \eta g_0^{-1}A_{+0}g_0\delta A_- - g_0^{-1}\delta A_+g_0\eta A_{-0} - g_0^{-1}\delta A_+g_0\delta A_- - g_0^{-1}A_{+0}g_0\eta\delta A_- \\ &+ \mu^2\big(\tfrac{1}{2}\eta^2g_0^{-1}Tg_0T + \tfrac{1}{2}g_0^{-1}Tg_0\eta^2T - \eta g_0^{-1}Tg_0\eta T\big) + \tfrac{1}{2}\delta\Psi_R[T,\partial_-\delta\Psi_R \\ &+ [A_{-0},\delta\Psi_R]] + \tfrac{1}{2}\delta\Psi_L[T,\partial_+\delta\Psi_L + [A_{+0},\delta\Psi_L]] + \mu g_0^{-1}\delta\Psi_Lg_0\delta\Psi_R\big]. \end{aligned} \tag{4.3}$$

For $\Psi_{R,L0} = 0$ the bosonic and fermionic fluctuations decouple at quadratic order; the fermionic sector describes only the physical fermionic degrees of freedom, that is the 16 real anticommuting fields parametrizing $\delta\Psi_R$ and $\delta\Psi_L$. Therefore, to determine the operator which acts on the fermions, we can simply extract it from the equations of motion for the fermionic fluctuations.

To isolate the physical bosonic fluctuations an H gauge needs to be fixed (or the unphysical fluctuations integrated out). The H gauge symmetry acts as follows (see (2.23))

$$\begin{aligned} &g_0e^\eta = g \to h^{-1}gh = h^{-1}g_0he^{h^{-1}\eta h}, \\ &A_{\pm0} + \delta A_\pm = A_\pm \to h^{-1}A_\pm h + h^{-1}\partial_\pm h = h^{-1}A_{\pm0}h + h^{-1}\partial_\pm h + h^{-1}\delta A_\pm h, \\ &\Psi_{R0} + \delta\Psi_R = \Psi_R \to h^{-1}\Psi_Rh = h^{-1}\Psi_{R0}h + h^{-1}\delta\Psi_Rh, \\ &\Psi_{L0} + \delta\Psi_L = \Psi_L \to h^{-1}\Psi_Lh = h^{-1}\Psi_{L0}h + h^{-1}\delta\Psi_Lh. \end{aligned} \tag{4.4}$$

These relations determine the transformations of the fluctuation fields and allow us to fix an H gauge.

4.2. Fluctuations near solutions in $AdS_2 \times S^2$ subspace of $AdS_5 \times S^5$

Let us consider the particular case of the expansion near the reduced theory solutions corresponding to the string theory solutions in the $\mathrm{AdS}_2 \times S^2$ subspace of $\mathrm{AdS}_5 \times S^5$. As discussed in appendix B.1, such reduced theory solutions can be parametrized as in (3.9) and (3.10). To fix the H gauge let us first note that we can always write $\eta^\perp$ and $\delta A_\pm$ as

$$\eta^\perp = \begin{pmatrix} u_1 + \sigma_3u_2^*\sigma_3 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & u_1^* - \sigma_3u_2\sigma_3 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & u_3 - \sigma_3u_4\sigma_3 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & u_3^* + \sigma_3u_4^*\sigma_3 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\delta A_+ = \begin{pmatrix} \mathrm{a}_{1+} - \mathrm{a}_{2+}^* & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathrm{a}_{1+}^* + \mathrm{a}_{2+} & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathrm{a}_{3+} + \mathrm{a}_{4+} & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathrm{a}_{3+}^* - \mathrm{a}_{4+}^* \end{pmatrix},$$

$$\delta A_- = \begin{pmatrix} \mathrm{a}_{1-} + \sigma_3\mathrm{a}_{2-}^*\sigma_3 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathrm{a}_{1-}^* - \sigma_3\mathrm{a}_{2-}\sigma_3 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathrm{a}_{3-} - \sigma_3\mathrm{a}_{4-}\sigma_3 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathrm{a}_{3-}^* + \sigma_3\mathrm{a}_{4-}^*\sigma_3 \end{pmatrix}.$$

Here u_i and $\mathrm{a}_{i\pm}$, $(i = 1, 2, 3, 4)$, are all elements of $\mathfrak{su}(2)$. The H gauge is then partially fixed by choosing

$$(\partial_- u_2 - \mathrm{a}_{2-})\partial_+\phi_\mathrm{A} + \tfrac{1}{2}\partial_-(\mathrm{a}_{2+}\sinh 2\phi_\mathrm{A}) = 0, \tag{4.5}$$

$$(\partial_- u_4 - \mathrm{a}_{4-})\partial_+\phi_\mathrm{S} + \tfrac{1}{2}\partial_-(\mathrm{a}_{4+}\sin 2\phi_\mathrm{S}) = 0. \tag{4.6}$$

This H gauge choice fixes 6 of the 12degrees of freedom of the H gauge symmetry. The reason for choosing this gauge is that when the classical solution and the gauge-fixing conditions are substituted into the quadratic fluctuation Lagrangian (4.3), the physical fluctuation fields, $\eta^{\parallel}$, decouple from the remaining unphysical fluctuation fields.

Now that the physical fluctuations have been decoupled we should be able to use the remaining H gauge symmetry to ensure that the sector of the Lagrangian containing the unphysical bosonic fluctuations does not produce a non-trivial contribution.

We are then left with the decoupled physical bosonic fluctuations. We may introduce the components of $\eta^{\parallel}$ as follows

$$\eta^{\parallel} = \begin{pmatrix} \eta^{\parallel}_A & 0 \\ 0 & \eta^{\parallel}_S \end{pmatrix}, \tag{4.7}$$

$$\eta_A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & \zeta_1 + i\zeta_2 & \zeta_3 + \mathrm{i}\zeta_4 \\ 0 & 0 & \zeta_3 - i\zeta_4 & -\zeta_1 + \mathrm{i}\zeta_2 \\ \zeta_1 - \mathrm{i}\zeta_2 & \zeta_3 + \mathrm{i}\zeta_4 & 0 & 0 \\ \zeta_3 - \mathrm{i}\zeta_4 & -\zeta_1 - \mathrm{i}\zeta_2 & 0 & 0 \end{pmatrix},$$

$$\eta_S = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & \zeta_5 + \mathrm{i}\zeta_6 & \zeta_7 + \mathrm{i}\zeta_8 \\ 0 & 0 & \zeta_7 - \mathrm{i}\zeta_8 & -\zeta_5 + \mathrm{i}\zeta_6 \\ \zeta_5 - \mathrm{i}\zeta_6 & \zeta_7 + \mathrm{i}\zeta_8 & 0 & 0 \\ \zeta_7 - \mathrm{i}\zeta_8 & -\zeta_5 - \mathrm{i}\zeta_6 & 0 & 0 \end{pmatrix}.$$

The corresponding part of the fluctuation Lagrangian is then

$$L_b = -\sum_{i=1}^{4} \zeta_i(\partial_+\partial_- + \mu^2\cosh 2\phi_A)\zeta_i - \sum_{i=5}^{8}\zeta_i(\partial_+\partial_- + \mu^2\cos 2\phi_S)\zeta_i. \tag{4.8}$$

Then the resulting bosonic part of the one-loop partition function is given by

$$([\det(\partial_+\partial_- + \mu^2\cosh 2\phi_A)\ \det(\partial_+\partial_- + \mu^2\cos 2\phi_S)]^4)^{-1/2}. \tag{4.9}$$

This is the same result as was found for the fluctuations of the conformal-gauge string theory around the classical solutions in the $\mathrm{AdS}_2 \times S^2$ subspace of $\mathrm{AdS}_5 \times S^5$ in section 3.2 (see (3.22) and (3.23)).

As was already mentioned, to determine the fermionic fluctuation operator we may just use the equations of motion arising from varying the quadratic fluctuation Lagrangian (4.3). It is easy to see that these give the same operator as found for the conformal-gauge string theory fermionic fluctuations, see (3.24).

We conclude that for the classical solutions of $\mathrm{AdS}_5 \times S^5$ string theory localized in $\mathrm{AdS}_2 \times S^2$, the one-loop partition functions computed in the string theory and in the reduced theory are the same.

4.3. Homogeneous string solutions

Let us now consider another class of $AdS_5 \times S^5$ string solutions—'homogeneous solutions' [23, 24, 26–28]—for which the string has a rigid shape and for which one can arrange to have the coefficients in the quadratic fluctuation Lagrangian to be constant. In this case the determinants of the operators which enter the one-loop partition function are expressed in terms of the characteristic frequencies which are relatively simple to calculate and compare between the conformal-gauge string theory and the reduced theory.

Our approach will be to start with a homogeneous solution of the conformal-gauge string theory and construct the corresponding field f using the parametrization of $PSU(2,2|4)$ in terms of the embedding coordinates as described in appendix B. Then the classical solution of the reduced theory will be found following the reduction procedure outlined in section 2. Since in the process of the reduction the natural $H \times H$ gauge symmetry of the string equations of motion is partially fixed to a H gauge symmetry, the solution of the reduced theory will correspond to the string theory solution in this partial gauge.

We use the H gauge symmetry to choose the classical solution of the reduced theory such that $g_0^{-1}\partial_\pm g_0$ and $g_0^{-1}Tg_0$ are constant. This is possible for the homogeneous solutions that we consider below (and should be possible in general). The reason for choosing this gauge is to help construct a Lagrangian for the physical fluctuations which has constant coefficients.

The quadratic fluctuation Lagrangian (4.3) can then be used to find the characteristic frequencies of fluctuations around the reduced theory solution. We will see that is possible to choose the H gauge on the fluctuations so that the coefficients in the quadratic fluctuation Lagrangian for the eight bosonic and eight fermionic physical fluctuations are all constant. It is then easy to compute the corresponding fluctuation frequencies. The resulting fluctuation frequencies around the classical solutions of the reduced theory will be shown to match the previously found frequencies of fluctuations around the homogeneous solutions in string theory.

4.3.1. Homogeneous string solution in $\mathbb{R}_t \times S^3$. One example of a simple string theory solution we shall consider here is the rigid circular two-spin string on S^3 in S^5 discussed in [23, 25, 27, 28]. Using the embedding coordinates in appendix B, i.e. Y_M $(M = -1,\ 0, \ldots,\ 4)$ of $\mathbb{R}^{4,2}$ for the AdS_5 part and X_I $(I = 1,\ 2,\ \ldots,\ 6)$ of $\mathbb{R}^6$ for the S^5 part, this bosonic string solution is

$$\begin{aligned} &Y_0 + \mathrm{i}Y_{-1} = \mathrm{e}^{\mathrm{i}\kappa\tau}, \qquad Y_1 = Y_2 = Y_3 = Y_4 = 0,\\ &X_1 + \mathrm{i}X_2 = \tfrac{1}{\sqrt{2}}\,\mathrm{e}^{\mathrm{i}\omega\tau + \mathrm{i}m\sigma}, \qquad X_3 + \mathrm{i}X_4 = \tfrac{1}{\sqrt{2}}\,\mathrm{e}^{\mathrm{i}\omega\tau - \mathrm{i}m\sigma}, \qquad X_5 = X_6 = 0. \end{aligned} \tag{4.10}$$

The Virasoro constraints imply that the three parameters, κ, ω and m, are related by

$$\kappa^2 = m^2 + \omega^2. \tag{4.11}$$

Using the parametrizations discussed in appendix B we obtain the corresponding bosonic coset element f,

$$f = \begin{pmatrix} f_A & \mathbf{0}_4 \\ \mathbf{0}_4 & f_S \end{pmatrix}$$

$$
f_A = \begin{pmatrix} e^{\frac{i\kappa\tau}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{i\kappa\tau}{2}} & 0 & 0 \\ 0 & 0 & e^{-\frac{i\kappa\tau}{2}} & 0 \\ 0 & 0 & 0 & e^{-\frac{i\kappa\tau}{2}} \end{pmatrix},
$$

$$
f_S = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i}{2}\,e^{-i\omega\tau+im\sigma} & -\frac{i}{2}\,e^{-i\omega\tau-im\sigma} \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{2}\,e^{i\omega\tau+im\sigma} & \frac{i}{2}\,e^{i\omega\tau-im\sigma} \\ \frac{i}{2}\,e^{i\omega\tau-im\sigma} & \frac{i}{2}\,e^{-i\omega\tau-im\sigma} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{2}\,e^{i\omega\tau+im\sigma} & \frac{i}{2}\,e^{-i\omega\tau+im\sigma} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{4.12}
$$

The corresponding solution of the reduced theory is[14]

$$
g_0 = \begin{pmatrix} g_A & \mathbf{0}_4 \\ \mathbf{0}_4 & g_S \end{pmatrix}, \qquad v \equiv e^{i\frac{\kappa^2-m^2}{\kappa}\tau}, \tag{4.13}
$$

$$
g_A = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \qquad g_S = \begin{pmatrix} 0 & \frac{\omega}{\kappa}v & -i\frac{m}{\kappa}v & 0 \\ -\frac{\omega}{\kappa}v^* & 0 & 0 & i\frac{m}{\kappa}v^* \\ i\frac{m}{\kappa}v & 0 & 0 & -\frac{\omega}{\kappa}v \\ 0 & -i\frac{m}{\kappa}v^* & \frac{\omega}{\kappa}v^* & 0 \end{pmatrix},
$$

$$
A_{+0} = \left(\begin{array}{c|cccc} \mathbf{0}_4 & & \mathbf{0}_4 & & \\ \hline & i\left(\frac{m^2}{\kappa}-\frac{\kappa}{2}\right) & 0 & 0 & 0 \\ \mathbf{0}_4 & 0 & -i\left(\frac{m^2}{\kappa}-\frac{\kappa}{2}\right) & 0 & 0 \\ & 0 & 0 & i\left(\frac{m^2}{\kappa}-\frac{\kappa}{2}\right) & 0 \\ & 0 & 0 & 0 & -i\left(\frac{m^2}{\kappa}-\frac{\kappa}{2}\right) \end{array}\right), \tag{4.14}
$$

$$
A_{-0} = \left(\begin{array}{c|cccc} \mathbf{0}_4 & & \mathbf{0}_4 & & \\ \hline & -i\frac{\kappa}{2} & 0 & 0 & 0 \\ \mathbf{0}_4 & 0 & i\frac{\kappa}{2} & 0 & 0 \\ & 0 & 0 & -i\frac{\kappa}{2} & 0 \\ & 0 & 0 & 0 & i\frac{\kappa}{2} \end{array}\right), \qquad \Psi_{R0} = \Psi_{L0} = 0. \tag{4.15}
$$

Note that the point-like string (BMN vacuum) solution is a particular case of (4.10), that is when $m = 0$ and $\omega = \kappa$. In the reduced theory the corresponding limit of (4.13) is a special case of the vacuum in (2.27)[15].

Since the classical fermionic fields vanish, the bosonic AdS_5 sector, the bosonic S^5 sector and the fermionic sector all decouple at the level of the action and we can discuss them separately.

Here the AdS_5 part of g_0 lives in H and is constant[16]. As discussed in section 2.2 this is a vacuum solution of this sector. The resulting fluctuation Lagrangian in the bosonic AdS_5 sector is

$$
\begin{aligned} L_A = \mathrm{STr}\big[&\tfrac{1}{2}\partial_+\eta\partial_-\eta - \delta A_-\partial_+\eta + \delta A_+ g_0\partial_-\eta g_0^{-1} + \delta A_+\delta A_- - g_0^{-1}\delta A_+ g_0\delta A_- \\ &+ \kappa^2(\eta\eta T^2 - \eta T\eta T)\big]. \end{aligned} \tag{4.16}
$$

[14] The μ parameter of the reduced theory here is identified as κ.

[15] One may also consider a formally different embedding of the string solution (4.10) into the reduced theory for which the point-like limit corresponds to the trivial vacuum $g = \mathbf{1}$. In this case the solution for g has σ instead of τ dependence, see appendix D.

[16] In the AdS_5 case we shall assume that the field is just in the top left 4×4 matrix of the original (8×8) field and similarly for the S^5 case the field will be just in the bottom right 4×4 matrix.

We partially fix the H gauge symmetry by setting the diagonal components of $\eta^\perp$ to zero[17]. After integrating out $\delta A_\pm$ the Lagrangian describing only the physical fluctuations is

$$L_A = \mathrm{STr}\left[\tfrac{1}{2}\partial_+\eta^\parallel\partial_-\eta^\parallel + \kappa^2(\eta^\parallel\eta^\parallel T^2 - \eta^\parallel T\eta^\parallel T)\right]. \tag{4.17}$$

Let us introduce the component fields of $\eta^\parallel$ as

$$\eta^\parallel = \begin{pmatrix} 0 & 0 & a_1+\mathrm{i}a_2 & a_3+\mathrm{i}a_4 \\ 0 & 0 & a_3-\mathrm{i}a_4 & -a_1+\mathrm{i}a_2 \\ a_1-\mathrm{i}a_2 & a_3+\mathrm{i}a_4 & 0 & 0 \\ a_3-\mathrm{i}a_4 & -a_1-\mathrm{i}a_2 & 0 & 0 \end{pmatrix}. \tag{4.18}$$

Then (4.17) becomes

$$L_A = 2\sum_{i=1}^{4}\left(\partial_+a_i\partial_-a_i - \kappa^2a_i^2\right), \tag{4.19}$$

which describes four bosonic fluctuations with frequency $\sqrt{n^2+\kappa^2}$, $n \in \mathbb{Z}$.

Now let us consider the S^5 sector. We introduce the following parametrization of $\eta^\parallel$, $\eta^\perp$ and $\delta A_\pm$,

$$\eta^\parallel = \begin{pmatrix} 0 & 0 & b_1+\mathrm{i}b_2 & b_3+\mathrm{i}b_4 \\ 0 & 0 & -b_3+\mathrm{i}b_4 & b_1-\mathrm{i}b_2 \\ -b_1+\mathrm{i}b_2 & b_3+\mathrm{i}b_4 & 0 & 0 \\ -b_3+\mathrm{i}b_4 & -b_1-\mathrm{i}b_2 & 0 & 0 \end{pmatrix}, \tag{4.20}$$

$$\eta^\perp = \begin{pmatrix} \mathrm{i}h_1 & h_2+\mathrm{i}h_3 & 0 & 0 \\ -h_2+\mathrm{i}h_3 & -\mathrm{i}h_1 & 0 & 0 \\ 0 & 0 & \mathrm{i}h_4 & h_5+\mathrm{i}h_6 \\ 0 & 0 & -h_5+\mathrm{i}h_6 & -\mathrm{i}h_4 \end{pmatrix}, \tag{4.21}$$

$$\begin{aligned} \delta A_+ &= \begin{pmatrix} \mathrm{i}a_{+1} & (a_{+2}+\mathrm{i}a_{+3})\,v^2 & 0 & 0 \\ -(a_{+2}-\mathrm{i}a_{+3})\,v^{*2} & -\mathrm{i}a_{+1} & 0 & 0 \\ 0 & 0 & \mathrm{i}a_{+4} & (a_{+5}+\mathrm{i}a_{+6})\,v^2 \\ 0 & 0 & -(a_{+5}-ia_{+6})\,v^{*2} & -\mathrm{i}a_{+4} \end{pmatrix}, \\ \delta A_- &= \begin{pmatrix} \mathrm{i}a_{-1} & a_{-2}+\mathrm{i}a_{-3} & 0 & 0 \\ -a_{-2}+\mathrm{i}a_{-3} & -\mathrm{i}a_{-1} & 0 & 0 \\ 0 & 0 & \mathrm{i}a_{-4} & a_{-5}+\mathrm{i}a_{-6} \\ 0 & 0 & -a_{-5}+\mathrm{i}a_{-6} & -\mathrm{i}a_{-4} \end{pmatrix}. \end{aligned} \tag{4.22}$$

When we substitute this into the bosonic part of the quadratic fluctuation Lagrangian (4.3), the fields decouple into two smaller sectors. These are, first, a sector containing b_3, b_4 and the diagonal components of $\eta^\perp$, $\delta A_\pm$, which has a Lagrangian with constant coefficients, and second, a sector containing b_1,b_2 and the off-diagonal components of $\eta^\perp$, $\delta A_\pm$. The coefficients in this sector have some τ dependence, arising from the $\delta A_+\delta A_-$ term (v defined in (4.13) depends on τ).

If the gauge field fluctuations are integrated out first, we end up with a Lagrangian that has τ-dependent coefficients. To avoid this complication, i.e. to construct an action containing only physical fluctuations and having constant coefficients we choose the following partial gauge fixing

$$\begin{aligned} &h_1 + h_4 = \mathrm{const}, \\ &\kappa(a_{-2}-a_{-5}) - \kappa^2(h_3-h_6) - \partial_-(a_{+3}-a_{+6}) - \kappa\partial_-(h_2-h_5) = 0, \\ &\kappa(a_{-3}+a_{-6}) + \kappa^2(h_2+h_5) - \partial_-(a_{+2}+a_{+5}) - \kappa\partial_-(h_3+h_6) = 0. \end{aligned} \tag{4.23}$$

[17] This is to completely remove the degeneracy of expanding around this vacuum.

Then we can easily integrate out the diagonal components of $\delta A_\pm$ to get a Lagrangian for b_3 and b_4 in the desired form. The second two gauge constraints are chosen to decouple b_1 and b_2 from the unphysical fluctuations. By using the remaining gauge freedom we should be able to ensure that the unphysical fields only give a trivial contribution to the partition function.

The resulting Lagrangian for this sector is then

$$L_S = 2\left[\sum_{i=1}^{4} \partial_- b_i \partial_+ b_i + \sum_{i=1}^{2}(2m^2 - \kappa^2)b_i^2 + 4m^2 b_4^2 + 2\kappa(b_4\partial_+ b_3 + b_4\partial_- b_3)\right]. \tag{4.24}$$

This Lagrangian describes two decoupled fluctuations, b_1, b_2, with frequencies

$$\sqrt{n^2 + \kappa^2 - 2m^2}, \qquad n \in \mathbb{Z}, \tag{4.25}$$

and two coupled fluctuations, b_3, b_4, with frequencies

$$\sqrt{n^2 + 2\kappa^2 - 2m^2 \pm 2\sqrt{n^2\kappa^2 + (m^2 - \kappa^2)^2}}, \qquad n \in \mathbb{Z}. \tag{4.26}$$

In appendix E we shall present an alternative way of computing these fluctuation frequencies which does not involve the above gauge fixing (4.23).

The fermionic sector is described by

$$\begin{aligned} L_{\text{ferm}} = \mathrm{STr}\big(&\tfrac{1}{2}\delta\Psi_R[T, \partial_-\delta\Psi_R + [A_{0-}, \delta\Psi_R]] \\ &+ \tfrac{1}{2}\delta\Psi_L[T, \partial_+\delta\Psi_L + [A_{0+}, \delta\Psi_L]] + \kappa g_0^{-1}\delta\Psi_L g_0 \delta\Psi_R\big). \end{aligned} \tag{4.27}$$

To make coefficients in this Lagrangian constant we may rotate some of the fermionic fields to cancel the contribution of g_0 and g_0^{-1} in the 'Yukawa' interaction term. This can be achieved by parametrizing the matrix components of $\delta\Psi_R$ and $\delta\Psi_L$ as follows

$$\delta\Psi_R = \begin{pmatrix} 0 & \mathfrak{X}_R \\ \mathfrak{Y}_R & 0 \end{pmatrix}, \qquad \delta\Psi_L = \begin{pmatrix} 0 & \mathfrak{X}_L \\ \mathfrak{Y}_L & 0 \end{pmatrix}, \tag{4.28}$$

where

$$\mathfrak{X}_R = \begin{pmatrix} 0 & 0 & \alpha_1 + \mathrm{i}\alpha_2 & \alpha_3 + \mathrm{i}\alpha_4 \\ 0 & 0 & -\alpha_3 + \mathrm{i}\alpha_4 & \alpha_1 - \mathrm{i}\alpha_2 \\ \alpha_5 + \mathrm{i}\alpha_6 & \alpha_7 - \mathrm{i}\alpha_8 & 0 & 0 \\ \alpha_7 + \mathrm{i}\alpha_8 & -\alpha_5 + \mathrm{i}\alpha_6 & 0 & 0 \end{pmatrix}, \tag{4.29}$$

$$\mathfrak{Y}_R = \begin{pmatrix} 0 & 0 & -\alpha_6 - \mathrm{i}\alpha_5 & -\alpha_8 - \mathrm{i}\alpha_7 \\ 0 & 0 & \alpha_8 - \mathrm{i}\alpha_7 & -\alpha_6 + \mathrm{i}\alpha_5 \\ \alpha_2 + \mathrm{i}\alpha_1 & \alpha_4 - \mathrm{i}\alpha_3 & 0 & 0 \\ \alpha_4 + \mathrm{i}\alpha_3 & -\alpha_2 + \mathrm{i}\alpha_1 & 0 & 0 \end{pmatrix}, \tag{4.30}$$

$$\mathfrak{X}_L = \begin{pmatrix} 0 & 0 & (\beta_1 + i\beta_2)v^* & (\beta_3 + \mathrm{i}\beta_4)v \\ 0 & 0 & (\beta_3 - \mathrm{i}\beta_4)v^* & (-\beta_1 + \mathrm{i}\beta_2)v \\ (\beta_5 + \mathrm{i}\beta_6)v^* & (-\beta_7 + \mathrm{i}\beta_8)v & 0 & 0 \\ (\beta_7 + \mathrm{i}\beta_8)v^* & (\beta_5 - \mathrm{i}\beta_6)v & 0 & 0 \end{pmatrix}, \tag{4.31}$$

$$\mathfrak{Y}_L = \begin{pmatrix} 0 & 0 & (-\beta_6 - \mathrm{i}\beta_5)v & (-\beta_8 - \mathrm{i}\beta_7)v \\ 0 & 0 & (-\beta_8 + \mathrm{i}\beta_7)v^* & (\beta_6 - \mathrm{i}\beta_5)v^* \\ (\beta_2 + \mathrm{i}\beta_1)v & (-\beta_4 + \mathrm{i}\beta_3)v & 0 & 0 \\ (\beta_4 + \mathrm{i}\beta_3)v^* & (\beta_2 - \mathrm{i}\beta_1)v^* & 0 & 0 \end{pmatrix}. \tag{4.32}$$

Here α_k and β_k are 8+8 real anti-commuting functions and υ is defined in (4.13). The Lagrangian (4.27) then takes the form

$$L_{\text{ferm}} = 2\Bigg[\sum_{i=1}^{8}(\alpha_i\partial_-\alpha_i + \beta_i\partial_+\beta_i) + \sqrt{\kappa^2+m^2}(-\alpha_1\alpha_2+\alpha_3\alpha_4-\alpha_5\alpha_6-\alpha_7\alpha_9+\beta_1\beta_2-\beta_3\beta_4+\beta_5\beta_6+\beta_7\beta_8) + \sqrt{\kappa^2-m^2}(\alpha_1\beta_3+\alpha_3\beta_1-\alpha_5\beta_7-\alpha_7\beta_5-\beta_2\alpha_4+\beta_4\alpha_2+\beta_6\alpha_8-\beta_8\alpha_6)\Bigg], \tag{4.33}$$

which describes eight fermionic fluctuations with 4+4 sets of the frequencies,

$$\begin{aligned}&\sqrt{n^2-m^2+\frac{5\kappa^2}{4}+\sqrt{\kappa^4+n^2\kappa^2-m^2\kappa^2}},\\ &\sqrt{n^2-m^2+\frac{5\kappa^2}{4}-\sqrt{\kappa^4+n^2\kappa^2-m^2\kappa^2}}, \qquad n\in\mathbb{Z}.\end{aligned} \tag{4.34}$$

The characteristic frequencies found above directly from the reduced theory action are exactly the same as found [25, 26] from the $\mathrm{AdS}_5\times S^5$ string theory action expanded near the solution (4.10).[18]

We conclude that expanding the superstring action near the homogeneous 2-spin solution in $\mathbb{R}_t\times S^3$ and expanding the reduced theory action near its counterpart in the reduced theory one finds the same set of characteristic frequencies and thus the same one-loop contribution to the respective partition functions.

4.3.2. Large spin limit of the folded spinning string in $AdS_3\times S^1$. As another example we shall consider the large spin limit of the solution for a folded string in AdS_5 with spin S [29] orbiting also in S^5 with momentum J [30]. As was noted in [19, 31], in the limit when $\mathcal{S}=\frac{S}{\sqrt{\lambda}}\to\infty$ with $\frac{J}{\sqrt{\lambda}\,\ln S}$ fixed this solution simplifies and becomes homogeneous. In terms of the embedding coordinates (see appendix B) it takes the form (cf (4.10))

$$\begin{aligned}&Y_0+\mathrm{i}Y_{-1}=\cosh(\ell\sigma)\,\mathrm{e}^{\mathrm{i}\kappa\tau}, && Y_1+\mathrm{i}Y_2=\sinh(\ell\sigma)\,\mathrm{e}^{\mathrm{i}\kappa\tau}, && Y_3=Y_4=0,\\ &X_1=X_2=X_3=X_4=0, && X_5+\mathrm{i}X_6=\mathrm{e}^{\mathrm{i}\nu\tau}, && \kappa^2=\ell^2+\nu^2,\end{aligned} \tag{4.35}$$

where it is assumed that $\kappa\sim\ell\gg 1$, and $\frac{\nu}{\kappa}$ is fixed (so that the closed-string periodicity condition in σ is satisfied asymptotically). This solution is, in fact, related to the $J_1=J_2$ solution in $\mathbb{R}_t\times S^3$ by a formal analytic continuation [19].

Using the parametrization in terms of the embedding coordinates discussed in appendix B we obtain the corresponding coset element f,

$$f=\begin{pmatrix} f_A & \mathbf{0}_4\\ \mathbf{0}_4 & f_S\end{pmatrix},$$

$$f_A=\begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}\kappa\tau}{2}}\cosh\frac{\ell\sigma}{2} & 0 & 0 & -\mathrm{e}^{\frac{3\mathrm{i}\kappa\tau}{2}}\sinh\frac{\ell\sigma}{2}\\ 0 & \mathrm{e}^{\frac{\mathrm{i}\kappa\tau}{2}}\cosh\frac{\ell\sigma}{2} & \mathrm{e}^{-\frac{\mathrm{i}\kappa\tau}{2}}\sinh\frac{\ell\sigma}{2} & 0\\ 0 & \mathrm{e}^{\frac{\mathrm{i}\kappa\tau}{2}}\sinh\frac{\ell\sigma}{2} & \mathrm{e}^{-\frac{\mathrm{i}\kappa\tau}{2}}\cosh\frac{\ell\sigma}{2} & 0\\ -\mathrm{e}^{-\frac{3\mathrm{i}\kappa\tau}{2}}\sinh\frac{\ell\sigma}{2} & 0 & 0 & \mathrm{e}^{-\frac{\mathrm{i}\kappa\tau}{2}}\cosh\frac{\ell\sigma}{2}\end{pmatrix}, \tag{4.36}$$

[18] Starting with the string solution in the form (4.10) used in [25] one finds that the fermions are naturally periodic [28].

$$f_S = \begin{pmatrix} e^{\frac{i\nu\tau}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{i\nu\tau}{2}} & 0 & 0 \\ 0 & 0 & e^{-\frac{i\nu\tau}{2}} & 0 \\ 0 & 0 & 0 & e^{-\frac{i\nu\tau}{2}} \end{pmatrix}.$$

The counterpart of this solution in the reduced theory is described by[19]

$$g_0 = \begin{pmatrix} g_A & \mathbf{0}_4 \\ \mathbf{0}_4 & g_S \end{pmatrix}, \qquad \upsilon \equiv e^{-i\frac{\kappa^2\tau}{\nu}}, \tag{4.37}$$

$$g_A = \begin{pmatrix} 0 & \frac{\kappa}{\nu}\upsilon & -\frac{\ell}{\nu}\upsilon & 0 \\ -\frac{\kappa}{\nu}\upsilon^* & 0 & 0 & \frac{\ell}{\nu}\upsilon^* \\ \frac{\ell}{\nu}\upsilon & 0 & 0 & -\frac{\kappa}{\nu}\upsilon \\ 0 & -\frac{\ell}{\nu}\upsilon^* & \frac{\kappa}{\nu}\upsilon^* & 0 \end{pmatrix}, \qquad g_S = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

$$A_{+0} = \begin{pmatrix} \begin{matrix} \frac{i\nu}{2} & 0 & 0 & 0 \\ 0 & -\frac{i\nu}{2} & 0 & 0 \\ 0 & 0 & \frac{i\nu}{2} & 0 \\ 0 & 0 & 0 & -\frac{i\nu}{2} \end{matrix} & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix}, \tag{4.38}$$

$$A_{-0} = \begin{pmatrix} \begin{matrix} \frac{i(m^2+\kappa^2)}{2\nu} & 0 & 0 & 0 \\ 0 & -\frac{i(\ell^2+\kappa^2)}{2\nu} & 0 & 0 \\ 0 & 0 & \frac{i(\ell^2+\kappa^2)}{2\nu} & 0 \\ 0 & 0 & 0 & -\frac{i(\ell^2+\kappa^2)}{2\nu} \end{matrix} & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix}.$$

$$\Psi_{R0} = \Psi_{L0} = 0. \tag{4.39}$$

Note that again the point-like string (BMN vacuum) solution is a particular case of (4.35), that is when $\ell = 0$ and $\nu = \kappa$. The corresponding limit of (4.37) is related by a simple H gauge transformation to a special case of the vacuum in (2.27).

This reduced theory background is very similar to the one in (4.13) corresponding to the homogeneous string solution in $\mathbb{R}_t \times S^3$. Carrying out a similar analysis of the quadratic fluctuation spectrum in the reduced theory action one finds the following bosonic

$$\begin{aligned} &1 \times \sqrt{n^2 + 2\kappa^2 + 2\sqrt{\kappa^4 + n^2\nu^2}}, \\ &1 \times \sqrt{n^2 + 2\kappa^2 - 2\sqrt{\kappa^4 + n^2\nu^2}}, \\ &2 \times \sqrt{n^2 + 2\kappa^2 - \nu^2}, \\ &4 \times \sqrt{n^2 + \nu^2} \end{aligned} \tag{4.40}$$

and fermionic

$$\begin{aligned} &4 \times \sqrt{n^2 + \kappa^2 + \frac{\nu^2}{4} + \sqrt{\nu^2(n^2 + \kappa^2)}}, \\ &4 \times \sqrt{n^2 + \kappa^2 + \frac{\nu^2}{4} - \sqrt{\nu^2(n^2 + \kappa^2)}} \end{aligned} \tag{4.41}$$

fluctuation frequencies. These are indeed exactly the same as following directly from the $AdS_5 \times S^5$ superstring action [31].

[19] Here the μ parameter of the reduced theory is identified as ν.

5. Concluding remarks

In this paper we discussed how to relate the semiclassical expansion in the original $AdS_5 \times S^5$ superstring theory (2.2) and the corresponding reduced theory (2.24). We considered several classes of string solutions, found their reduced model counterparts and then verified that the respective spectra of quadratic fluctuations match. This implies the matching of the one-loop partition functions (1.1).

Given that the classical equations (and their solutions) in the string theory and in the reduced theory are closely related, one may, of course, expect the quadratic fluctuations to match as well. However, this matching is still rather non-trivial given that one needs to partially fix the $H \times H$ gauge symmetry of the string equations written in terms of the reduced theory variables (2.20) in order to be able to construct a local Lagrangian of the reduced theory. One of the remaining open questions is if the reduced Lagrangians obtained via different gauge fixings (in particular, those parametrized by an automorphism τ, see (2.22), (2.25)) are actually equivalent at the quantum level.

It would be interesting to understand the equivalence between the corresponding quadratic fluctuation spectra in the string theory and in the reduced theory using their closely connected integrable structures. Indeed, fluctuation frequencies near particular finite gap solutions can be found directly from the corresponding algebraic curve description (see, e.g., [32]).

Another important open problem is to find out if the one-loop matching between the string and the reduced theory partition functions extends to the two-loop level. If it does, that would be a truly non-trivial confirmation of our conjecture (1.1). On the string theory side, the two-loop computation of the partition function was done for the infinite spin (or 'homogeneous') limit of the folded string solution [19, 20]. What remains is to compute the two-loop correction starting with the reduced theory action (2.24) and expanding it near the corresponding solution (4.37) and (4.38). Since the analysis of quadratic fluctuations on the reduced theory side is generally simpler than on the string theory side we expect that this two-loop computation may not be too complicated (cf also [4]).

Finally, as a step towards a solution of the reduced theory based on its integrability it remains to compute the corresponding 2D Lorentz-invariant massive S-matrix for the elementary excitations above the 'trivial' vacuum. There are technical complications when this is done directly by starting with the reduced theory based on the symmetrically gauged ($\tau = \mathbf{1}$) WZW model (2.25) expanded near the vacuum $g = \mathbf{1}$. However, one may try to expand near other vacua like (2.27) or consider a reduced model with a non-trivial automorphism τ (expecting still that the S-matrix should not depend on a choice of τ).

Acknowledgments

We acknowledge useful discussions with A Alexandrov, M Grigoriev, R Roiban and A Tirziu. AAT is particularly grateful to R Roiban for an initial collaboration on the issues addressed in this paper and numerous discussions of related questions. BH would like to thank EPSRC for his studentship. YI is supported by the Ishizaka Foundation. Part of this work was done while AAT was a participant of the 2009 program 'Fundamental Aspects of Superstring Theory' at the Kavli Institute for Theoretical Physics at Santa Barbara. AAT also acknowledges the hospitality of the Galileo Galilei Institute in Florence during the 2009 program 'Non-Perturbative Methods in Strongly Coupled Gauge Theories'.

Appendix A. $PSU(2,2|4)$: some definitions and notation

Here we will present a particular matrix representation of $PSU(2,2|4)$ which we used in the main text (see also [21, 1]). In particular, we shall make explicit the identification of the $\mathfrak{g} = \mathfrak{sp}(2,\ 2) \times \mathfrak{sp}(4)$ subalgebra whose corresponding group G is the subgroup G in the F/G coset sigma model, and also the group G in the G/H gauged WZW model.

Let us define the following matrices

$$\boldsymbol{\Sigma} = \begin{pmatrix} \Sigma & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{1}_4 \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} K & \mathbf{0}_4 \\ \mathbf{0}_4 & K \end{pmatrix}, \qquad \Sigma^2 = \mathbf{1}_4, \qquad K^2 = -\mathbf{1}_4, \tag{A.1}$$

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We can then write a generic element of the algebra $\mathfrak{psu}(2,2|4)$ as follows

$$\mathfrak{f} = \begin{pmatrix} \mathfrak{A} & \mathfrak{X} \\ \mathfrak{Y} & \mathfrak{B} \end{pmatrix}, \tag{A.2}$$

$$\mathfrak{f} = -\boldsymbol{\Sigma}^{-1}\mathfrak{f}^\dagger\boldsymbol{\Sigma}, \qquad \mathrm{Tr}\,\mathfrak{A} = \mathrm{Tr}\,\mathfrak{B} = 0,$$

$$\mathfrak{f}^\dagger = \begin{pmatrix} \mathfrak{A}^\dagger & -i\mathfrak{Y}^\dagger \\ -\mathrm{i}\mathfrak{X}^\dagger & \mathfrak{B}^\dagger \end{pmatrix}.$$

Here $\mathfrak{A}$ and $\mathfrak{B}$ are 4×4 matrices whose components are commuting while $\mathfrak{X}$ and $\mathfrak{Y}$ are 4×4 matrices whose components are anticommuting. We then have the following conditions on $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{X}$ and $\mathfrak{Y}$,

$$\Sigma\mathfrak{A}^\dagger\Sigma = -\mathfrak{A}, \qquad \mathfrak{B}^\dagger = -\mathfrak{B}, \qquad \mathrm{i}\Sigma\mathfrak{Y}^\dagger = \mathfrak{X}, \qquad \mathrm{i}\mathfrak{X}^\dagger\Sigma = \mathfrak{Y}. \tag{A.3}$$

Thus $\mathfrak{A} \in \mathfrak{su}(2,\ 2)$ and $\mathfrak{B} \in \mathfrak{su}(4)$. We can then decompose $\mathfrak{f}$ under a $\mathbb{Z}_4$ grading as follows

$$\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3, \tag{A.4}$$

$$-\mathbf{K}^{-1}\mathfrak{f}_r^{st}\mathbf{K} = i^r\mathfrak{f}_r, \qquad \mathfrak{f}_r^{st} = \begin{pmatrix} \mathfrak{A}^t & -\mathfrak{Y}^t \\ \mathfrak{X}^t & \mathfrak{B} \end{pmatrix}.$$

It is possible to write generic elements of $\mathfrak{f}_{0,2}$ as

$$\mathfrak{f}_{0,2} = \begin{pmatrix} \mathfrak{A}_{0,2} & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathfrak{B}_{0,2} \end{pmatrix}, \tag{A.5}$$

$$\mathfrak{A}_0 = K\mathfrak{A}_0^t K, \qquad \mathfrak{B}_0 = K\mathfrak{B}_0^t K,$$
$$\mathfrak{A}_2 = -K\mathfrak{A}_2^t K, \qquad \mathfrak{B}_2 = -K\mathfrak{B}_2^t K,$$

and generic elements of $\mathfrak{f}_{1,3}$ as

$$\mathfrak{f}_{1,3} = \begin{pmatrix} \mathbf{0}_4 & \mathfrak{X}_{1,3} \\ \mathfrak{Y}_{1,3} & \mathbf{0}_4 \end{pmatrix}, \tag{A.6}$$

$$\mathrm{i}\mathfrak{X}_1 = -K\mathfrak{Y}_1^t K, \qquad \mathrm{i}\mathfrak{X}_3 = K\mathfrak{Y}_3^t K.$$

The subspaces of this decomposition satisfy the following commutation relations

$$[\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{i+j \bmod 4}. \tag{A.7}$$

We identify $\mathfrak{f}_0 = \mathfrak{g}$ and $\mathfrak{f}_2 = \mathfrak{p}$. Then $\mathfrak{g}$ forms a subalgebra, and it is this algebra whose corresponding group is the group G in the F/G coset sigma model and in the G/H gauged WZW model.

It is now possible to perform a further $\mathbb{Z}_2$ decomposition, which allows us to define the group H in the G/H gauged WZW model. To do this we identify the following fixed element $T \in \mathfrak{f}_2$

$$T = \frac{\mathrm{i}}{2}\,\mathrm{diag}(1, 1, -1, -1, 1, 1, -1, -1). \tag{A.8}$$

The $\mathbb{Z}_2$ decomposition is then given by

$$\mathfrak{f}_r^{\parallel} = -[T, [T, \mathfrak{f}_r]], \qquad \mathfrak{f}_r^{\perp} = -\{T, \{T, \mathfrak{f}_r\}\}. \tag{A.9}$$

It should be noted that this is an orthogonal decomposition, that is

$$\begin{aligned} &\mathfrak{f} = \mathfrak{f}^{\parallel} \oplus \mathfrak{f}^{\perp}, \\ &\mathrm{STr}(\mathfrak{f}^{\parallel}\mathfrak{f}^{\perp}) = 0. \end{aligned} \tag{A.10}$$

Then

$$[\mathfrak{f}^{\perp}, \mathfrak{f}^{\perp}] \subset \mathfrak{f}^{\perp}, \qquad [\mathfrak{f}^{\perp}, \mathfrak{f}^{\parallel}] \subset \mathfrak{f}^{\parallel}, \qquad [\mathfrak{f}^{\parallel}, \mathfrak{f}^{\parallel}] \subset \mathfrak{f}^{\perp}. \tag{A.11}$$

We identify $\mathfrak{h} = \mathfrak{f}_0^{\perp}$, $\mathfrak{m} = \mathfrak{f}_0^{\parallel}$, $\mathfrak{a} = \mathfrak{f}_2^{\perp}$, $\mathfrak{n} = \mathfrak{f}_2^{\parallel}$. Elements from these subspaces satisfy

$$\begin{aligned} &[\mathfrak{a}, \mathfrak{a}] \subset 0, \qquad [\mathfrak{a}, \mathfrak{h}] \subset 0, \qquad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}, \\ &[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{a}] \subset \mathfrak{n}, \qquad [\mathfrak{n}, \mathfrak{a}] \subset \mathfrak{m} \end{aligned} \tag{A.12}$$

Here $\mathfrak{h}$ is a subalgebra; the corresponding subgroup is then identified as the group H in the G/H gauged WZW model. It is possible to show that $\mathfrak{h}$ has the following form

$$\begin{pmatrix} \mathfrak{h}_1 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathfrak{h}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathfrak{h}_3 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathfrak{h}_4 \end{pmatrix}, \tag{A.13}$$

where each $\mathfrak{h}_i$ is a copy of $\mathfrak{su}(2)$, i.e. $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathfrak{so}(4) \oplus \mathfrak{so}(4)$.

Finally as discussed in [21], it is possible to use the κ-symmetry to choose fermionic currents to take the form,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & 0 & \bullet & \bullet & 0 & 0 \\ 0 & 0 & \bullet & \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.14}$$

This is exactly the same as the structure of the fermionic elements of the $^{\parallel}$ space. Thus it is always possible to choose the κ-symmetry gauge such that the fermionic currents live in the $^{\parallel}$ space.

Appendix B. Parametrization in terms of embedding coordinates

Here we shall discuss the relation between the embedding coordinates in $AdS_5 \times S^5$ and parametrization of the corresponding $PSU(2,2|4)$ coset elements (see [21] for details).

Let us define six real coordinates Y^M on $\mathbb{R}^{4,2}$ ($M = -1, 0, \ldots, 4$) and six real coordinates X^I on $\mathbb{R}^6$ ($I = 1, 2, \ldots, 6$). To define AdS_5 and S^5 embedded in $\mathbb{R}^{4,2}$ and $\mathbb{R}^6$ we impose

$$\begin{aligned} &\eta^{4,2}_{MN} Y^M Y^N = -1, && \eta^{6,0}_{IJ} X^I X^J = 1, \\ &\eta^{4,2} = \mathrm{diag}(-1,\ -1,\ 1,\ 1,\ 1,\ 1), && \eta^{6,0} = \mathrm{diag}(1,\ 1,\ 1,\ 1,\ 1,\ 1). \end{aligned} \tag{B.1}$$

Finally we define another set of coordinates, $t,\ y_i$ on AdS_5 and $\theta,\ x_i$ on S^5, $i = 1,\ 2,\ 3,\ 4$:

$$Y^1 + \mathrm{i}Y^2 = \frac{y_1 + \mathrm{i}y_2}{1 - \frac{y^2}{4}}, \qquad Y^3 + iY^4 = \frac{y_3 + \mathrm{i}y_4}{1 - \frac{y^2}{4}}, \tag{B.2}$$

$$\begin{aligned} &Y^0 + \mathrm{i}Y^{-1} = \frac{1 + \frac{y^2}{4}}{1 - \frac{y^2}{4}}\,\mathrm{e}^{\mathrm{i}t}, \\ &X^1 + \mathrm{i}X^2 = \frac{x_1 + \mathrm{i}x_2}{1 + \frac{x^2}{4}}, \qquad X^3 + \mathrm{i}X^4 = \frac{x_3 + \mathrm{i}x_4}{1 + \frac{x^2}{4}}, \\ &X^5 + \mathrm{i}X^6 = \frac{1 - \frac{x^2}{4}}{1 + \frac{x^2}{4}}\,\mathrm{e}^{\mathrm{i}\theta}. \end{aligned} \tag{B.3}$$

Here $y^2 = y_i y_i$ and $x^2 = x_i x_i$. The corresponding metrics of AdS_5 and S^5 in terms of $t,\ y_i,\ \theta,\ x_i$ are

$$\begin{aligned} \eta^{4,2}_{MN}\,\mathrm{d}Y^M\,\mathrm{d}Y^N &= -\left(\frac{1 + \frac{y^2}{4}}{1 - \frac{y^2}{4}}\right)^2 \mathrm{d}t^2 + \frac{\mathrm{d}y_i\,\mathrm{d}y_i}{\left(1 - \frac{y^2}{4}\right)^2}, \\ \eta^{6,0}_{IJ}\,\mathrm{d}X^I\,\mathrm{d}X^J &= \left(\frac{1 - \frac{x^2}{4}}{1 + \frac{x^2}{4}}\right)^2 \mathrm{d}\theta^2 + \frac{\mathrm{d}x_i\,\mathrm{d}x_i}{\left(1 + \frac{x^2}{4}\right)^2}. \end{aligned} \tag{B.4}$$

A suitable choice of a bosonic coset element would be such that $\mathrm{STr}(f^{-1}\mathrm{d}f)^2$ coincides with the sum of the two metrics in (B.4). This allows us to relate the embedding coordinates with the bosonic coset element directly:

$$\begin{aligned} f &= \begin{pmatrix} f_A & \mathbf{0}_4 \\ \mathbf{0}_4 & f_S \end{pmatrix} \\ &= \begin{pmatrix} \exp\left(\frac{\mathrm{i}}{2} t\gamma_5\right) & \mathbf{0}_4 \\ \mathbf{0}_4 & \exp\left(\frac{\mathrm{i}}{2}\theta\gamma_5\right) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-\frac{y^2}{4}}}\left(\mathbf{1}_4 + \frac{1}{2} y_i\gamma_i\right) & \mathbf{0}_4 \\ \mathbf{0}_4 & \frac{1}{\sqrt{1+\frac{x^2}{4}}}\left(\mathbf{1}_4 + \frac{\mathrm{i}}{2} x_i\gamma_i\right) \end{pmatrix}. \end{aligned} \tag{B.5}$$

Here γ_k are the $\mathfrak{so}(5)$ Dirac matrices chosen as

$$\begin{aligned} &\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ &\gamma_4 = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned} \tag{B.6}$$

B.1. $AdS_2 \times S^2$

Let us now consider a special case of an $\mathrm{AdS}_2 \times S^2$ subspace of $\mathrm{AdS}_5 \times S^5$:

$$\begin{aligned}
&-(Y^{-1})^2 - (Y^0)^2 + (Y^1)^2 = -1,\\
&(X^1)^2 + (X^5)^2 + (X^6)^2 = 1,\\
&Y_2 = Y_3 = Y_4 = y_1 = y_3 = y_4 = 0,\\
&X_2 = X_3 = X_4 = x_1 = x_3 = x_4 = 0.
\end{aligned} \tag{B.7}$$

The explicit coordinates on $\mathrm{AdS}_2 \times S^2$ are $t,\ y = y_1,\ \theta,\ x = x_1$.

The corresponding parametrization of the $PSU(2, 2|4)$ element (B.5) is then

$$\begin{aligned}
f_A &= \sqrt{1-\tfrac{y^2}{4}}\begin{pmatrix} e^{\frac{it}{2}} & 0 & 0 & \frac{iy}{2}e^{\frac{it}{2}} \\ 0 & e^{\frac{it}{2}} & \frac{iy}{2}e^{\frac{it}{2}} & 0 \\ 0 & -\frac{iy}{2}e^{-\frac{it}{2}} & e^{-\frac{it}{2}} & 0 \\ -\frac{iy}{2}e^{-\frac{it}{2}} & 0 & 0 & e^{-\frac{it}{2}} \end{pmatrix},\\
f_S &= \sqrt{1+\tfrac{x^2}{4}}\begin{pmatrix} e^{\frac{i\theta}{2}} & 0 & 0 & -\frac{x}{2}e^{\frac{i\theta}{2}} \\ 0 & e^{\frac{i\theta}{2}} & -\frac{x}{2}e^{\frac{i\theta}{2}} & 0 \\ 0 & \frac{x}{2}e^{-\frac{i\theta}{2}} & e^{-\frac{i\theta}{2}} & 0 \\ \frac{x}{2}e^{-\frac{i\theta}{2}} & 0 & 0 & e^{-\frac{i\theta}{2}} \end{pmatrix}.
\end{aligned} \tag{B.8}$$

Following the prescription of Pohlmeyer reduction as discussed in section 2, we can make a G gauge transformation, $f_b \to f_b g'$, such that $\left(f_b^{-1}\partial_+ f_b\right)_{\mathfrak{p}} \in \mathfrak{a}$. We can then use the remaining conformal diffeomorphism invariance to set $\left(f_b^{-1}\partial_+ f_b\right)_{\mathfrak{p}} = \mu_+ T$. In terms of the embedding coordinates this then implies

$$\begin{aligned}
&-(\partial_+ Y^{-1})^2 - (\partial_+ Y^0)^2 + (\partial_+ Y^1)^2 = -\mu_+^2,\\
&(\partial_+ X^1)^2 + (\partial_+ X^5)^2 + (\partial_+ X^6)^2 = \mu_+^2.
\end{aligned} \tag{B.9}$$

The next step of the reduction is to find a element g_0 of G such that $\left(f_b^{-1}\partial_- f_b\right)_{\mathfrak{p}} = \mu_- g_0^{-1} T g_0$. The following element of G satisfies this relation

$$g_0 = \begin{pmatrix} g_A & \mathbf{0}_4 \\ \mathbf{0}_4 & g_S \end{pmatrix}, \tag{B.10}$$

$$g_A = \begin{pmatrix} i\cosh\phi_A & 0 & 0 & \sinh\phi_A \\ 0 & -i\cosh\phi_A & \sinh\phi_A & 0 \\ 0 & \sinh\phi_A & i\cosh\phi_A & 0 \\ \sinh\phi_A & 0 & 0 & -i\cosh\phi_A \end{pmatrix},$$

$$g_S = \begin{pmatrix} i\cos\phi_S & 0 & 0 & i\sin\phi_S \\ 0 & -i\cos\phi_S & i\sin\phi_S & 0 \\ 0 & i\sin\phi_S & i\cos\phi_S & 0 \\ i\sin\phi_S & 0 & 0 & -i\cos\phi_S \end{pmatrix},$$

provided the following relations are satisfied

$$\begin{aligned}
&-(\partial_- Y^{-1})^2 - (\partial_- Y^0)^2 + (\partial_- Y^1)^2 = -\mu_-^2,\\
&(\partial_- X^1)^2 + (\partial_- X^5)^2 + (\partial_- X^6)^2 = \mu_-^2,
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
&-\partial_+ Y^{-1}\partial_- Y^{-1} - \partial_+ Y^0\partial_- Y^0 + \partial_+ Y^1\partial_- Y^1 = -\mu^2\cosh 2\phi_A,\\
&\partial_+ X^1\partial_- X^1 + \partial_+ X^5\partial_- X^5 + \partial_+ X^6\partial_- X^6 = \mu^2\cos 2\phi_S,\\
&\mu^2 = \sqrt{\mu_+^2\mu_-^2}.
\end{aligned} \tag{B.12}$$

It is possible to check that the corresponding gauge fields $A_\pm$ in (2.22) vanish in this case.

Appendix C. Fluctuations near $AdS_2 \times S^2$ solutions: special cases

In section 3.2 it was shown that for a classical solution in $AdS_2 \times S^2$ the bosonic fluctuation equations are

$$\begin{aligned} &\partial_+\partial_-\zeta_i + \mu^2 \cosh 2\phi_A\ \zeta_i = 0, \qquad i = 1, \ldots, 4, \\ &\partial_+\partial_-\zeta_i + \mu^2 \cos 2\phi_S\ \zeta_i = 0, \qquad i = 5, \ldots, 8, \end{aligned} \tag{C.1}$$

and the fermionic fluctuation equations are given by the following sets of coupled equations

$$\begin{aligned} &\partial_-\vartheta_i + \mu \cos\phi_S \cosh\phi_A \vartheta'_i + \mu \sin\phi_S \sinh\phi_A \vartheta'_{i+1} = 0, \\ &\partial_+\vartheta'_i - \mu \cos\phi_S \cosh\phi_A \vartheta_i + \mu \sin\phi_S \sinh\phi_A \vartheta_{i+1} = 0, \qquad i = 1, 3, 5, 7 \\ &\partial_-\vartheta_{i+1} + \mu \cos\phi_S \cosh\phi_A \vartheta'_{i+1} - \mu \sin\phi_S \sinh\phi_A \vartheta'_i = 0, \\ &\partial_+\vartheta'_{i+1} - \mu \cos\phi_S \cosh\phi_A \vartheta_{i+1} - \mu \sin\phi_S \sinh\phi_A \vartheta_i = 0. \end{aligned} \tag{C.2}$$

Below we shall consider some special cases of these equations.

C.1. Giant magnon

Here we shall check the general claim that the above equations give the same one-loop correction as the calculation following directly from the string theory action written in terms of coordinates on $AdS_5 \times S^5$ with the example of the giant magnon solution [10, 33]. For the giant magnon string solution we decompactify the spatial world-sheet direction (the energy and angular momentum of the string are taken to infinity). Its counterpart in the reduced theory is the vacuum and kink solutions of the sinh-Gordon and sine-Gordon equations respectively

$$\phi_A = 0, \qquad \phi_S = 2 \arctan \mathrm{e}^{\frac{\sigma - v\tau}{\sqrt{1-v^2}}}. \tag{C.3}$$

When taking the large energy/spin limit we rescale the world-sheet coordinates by μ and then send $\mu \to \infty$. As a result, μ scales out of the fluctuation equations (C.1) and (C.2). We may also change to the Lorentz-boosted coordinates

$$\Sigma = \frac{\sigma - v\tau}{\sqrt{1-v^2}}, \qquad \mathcal{T} = \frac{\sigma - v\tau}{\sqrt{1-v^2}}. \tag{C.4}$$

The bosonic AdS_5 fluctuation equations are given by four copies of

$$\partial_+\partial_-\zeta_A + \zeta_A = 0. \tag{C.5}$$

The bosonic S fluctuation equations are given by four copies of

$$\partial_+\partial_-\zeta_S + (1 - 2\,\mathrm{sech}^2\Sigma)\zeta_S = 0. \tag{C.6}$$

As discussed in [33] to compute the one-loop fluctuation operator determinant for the giant magnon string solution we should first look for the plane-wave solutions of the fluctuation equations. The plane-wave solutions of (C.5) are proportional to

$$\zeta_A = \mathrm{e}^{\mathrm{i}k\Sigma + \mathrm{i}\omega\mathcal{T}}, \qquad \omega^2 = k^2 + 1, \tag{C.7}$$

and of (C.6) to

$$\zeta_S = \mathrm{e}^{\mathrm{i}k\Sigma + \mathrm{i}\omega\mathcal{T}}(\tanh\Sigma + \mathrm{i}k), \qquad \omega^2 = k^2 + 1. \tag{C.8}$$

Finally, the fermionic fluctuation equations are given by eight copies of

$$\begin{aligned} &\partial_- \vartheta - \tanh \Sigma \vartheta' = 0, \\ &\partial_+ \vartheta' + \tanh \Sigma \vartheta = 0. \end{aligned} \tag{C.9}$$

After some simple maipulation with expressions in [33] it is easy to see that this system has plane-wave solutions proportional to

$$\begin{aligned} &\vartheta = -\frac{(1-v)|\omega - k|}{\sqrt{1-v^2}} \mathrm{e}^{\mathrm{i}k\Sigma - \mathrm{i}\omega\mathcal{T}} \mathrm{e}^{\frac{\mathrm{i}}{2}(\arctan(-\omega\sinh 2\Sigma) - \arctan(k\tanh 2\Sigma))} \operatorname{sech}\Sigma\sqrt{|\omega\cosh 2\Sigma - k|}, \\ &\vartheta' = \mathrm{e}^{\mathrm{i}k\Sigma - \mathrm{i}\omega\mathcal{T}} \mathrm{e}^{\frac{\mathrm{i}}{2}(\arctan(\omega\sinh 2\Sigma) - \arctan(k\tanh 2\Sigma))} \operatorname{sech}\Sigma\sqrt{|\omega\cosh 2\Sigma + k|}, \\ &\omega^2 = k^2 + 1. \end{aligned} \tag{C.10}$$

Following [33] we may then compute the stability angles for these solutions. To do this we put the system in a box of length $L \gg 1$, with $\sigma \sim \sigma + L$. From the form of the classical solution the system is also periodic in time with period $T_p = \frac{L}{v}$. The stability angle ν of an arbitrary fluctuation $\delta\phi$ is defined to be

$$\delta\phi(\tau + T_p, \sigma) = \mathrm{e}^{-\mathrm{i}\nu}\delta\phi(\tau, \sigma). \tag{C.11}$$

From (C.7) the four stability angles from the bosonic AdS_5 sector are

$$\nu_k(\zeta_A) = \frac{L}{v}\frac{\omega + vk}{\sqrt{1-v^2}}. \tag{C.12}$$

From (C.8) the four stability angles from the bosonic S^5 sector are

$$\nu_k(\zeta_S) = \frac{L}{v}\frac{\omega + vk}{\sqrt{1-v^2}} + 2\cot^{-1} k. \tag{C.13}$$

Finally, from (C.10) the eight stability angles from the fermionic sector are

$$\nu_k(\vartheta, \vartheta') = \frac{L}{v}\frac{\omega + vk}{\sqrt{1-v^2}} + \cot^{-1} k. \tag{C.14}$$

These agree exactly with the results in [33] derived directly by considering fluctuations of coordinates on $\mathrm{AdS}_5 \times S^5$. We then reproduce the final result of [33] that the sum over the stability angles (with a negative sign for the fermionic contribution) vanishes and thus so does the one-loop correction to the logarithm of the partition function or the energy of the giant magnon.

C.2. Some other examples

Here we briefly consider some other interesting solutions in $\mathrm{AdS}_2 \times S^2$. As discussed in [13] the reduced theory solutions

$$\phi_S = \mathrm{am}\left(\frac{\mu\,(\tau - v\sigma)}{k\sqrt{1-v^2}}, k^2\right), \qquad \phi_A = 0, \tag{C.15}$$

and

$$\phi_S = \frac{\pi}{2} + \mathrm{am}\left(\frac{\mu\,(\sigma - v\tau)}{k\sqrt{1-v^2}}, k^2\right), \qquad \phi_A = 0, \tag{C.16}$$

give rise to single-spin helical strings, effectively living on $\mathbb{R}_t \times S^2$.[20]

These solutions include some special cases. For example if we take the $v \to 0$ limit in (C.15) the corresponding string solution is a string pulsating on S^2, which is also discussed in

[20] am is the Jacobi amplitude function.

[36]. If we take the $k \to \infty$, $\mu \to \infty$, $\frac{\mu}{k} \to 1$ limit of (C.16) we get the sine-Gordon kink solution, which, as previously discussed, corresponds to the giant magnon string solution [10].

For both (C.15) and (C.16) the bosonic fluctuation equations from the AdS_5 sector are trivial, as we just have the vacuum solution. For the S^5 sector we obtain four copies of the following equations:

$$\left(\partial_+\partial_- + \mu^2\left[2\mathrm{cn}^2\left(\frac{\mu\,(\tau - v\sigma)}{k\sqrt{1-v^2}}, k^2\right) - 1\right]\right)\zeta_S = 0 \tag{C.17}$$

for (C.15) and

$$\left(\partial_+\partial_- + \mu^2\left[1 - 2\mathrm{cn}^2\left(\frac{\mu\,(\sigma - v\tau)}{k\sqrt{1-v^2}}, k^2\right)\right]\right)\zeta_S = 0. \tag{C.18}$$

for (C.16). These are strongly related to the $n = 1$ Lamé equation, [37]. For the fermionic sector we obtain eight copies of the following coupled systems

$$\partial_-\vartheta + \mu\,\mathrm{cn}\left(\frac{\mu\,(\tau - v\sigma)}{k\sqrt{1-v^2}}, k^2\right)\vartheta' = 0 \qquad \partial_+\vartheta' - \mu\,\mathrm{cn}\left(\frac{\mu\,(\tau - v\sigma)}{k\sqrt{1-v^2}}, k^2\right)\vartheta = 0 \tag{C.19}$$

for (C.15) and

$$\partial_-\vartheta - \mu\,\mathrm{sn}\left(\frac{\mu\,(\sigma - v\tau)}{k\sqrt{1-v^2}}, k^2\right)\vartheta' = 0 \qquad \partial_+\vartheta' + \mu\,\mathrm{sn}\left(\frac{\mu\,(\sigma - v\tau)}{k\sqrt{1-v^2}}, k^2\right)\vartheta = 0 \tag{C.20}$$

for (C.16). In various special cases the spectra and determinants of these operators have been studied in much detail [37–39]. Therefore, it should be possible to compute the corresponding one-loop correction to the logarithm of the partition function at least numerically.

Appendix D. Examples of reduced theory counterparts of some simple string solutions

Here we shall consider the reduced theory counterparts of the homogeneous string solutions on $\mathbb{R}_t \times S^3$ and $AdS_3 \times S^1$. Compared to the discussion in section 4.3 we shall assume the trivial embedding of these solutions into the reduced theory when it can be truncated to the complex sine-Gordon or complex sinh-Gordon models, respectively. The bosonic part of the reduced theory counterpart of $AdS_3 \times S^3$ string theory is described by ($\partial_\pm = \partial_\tau \pm \partial_\sigma$),[21]

$$L_B = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\,\partial_+\theta\partial_-\theta + \partial_+\phi\partial_-\phi + \coth^2\phi\,\partial_+\chi\partial_-\chi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi). \tag{D.1}$$

A particular simple solution of the resulting equations of motion is[22]

$$\varphi = \varphi_0, \qquad \phi = \phi_0, \qquad \theta = n\sigma + a\tau, \qquad \chi = k\sigma + b\tau\,, \tag{D.2}$$

$$\mu^2\sin^4\varphi_0 = n^2 - a^2, \qquad \mu^2\sinh^4\phi_0 = k^2 - b^2. \tag{D.3}$$

In the case of the $J_1 = J_2$ homogeneous string solution in $\mathbb{R}_t \times S^3$ (4.10) we have

$$t = \kappa\tau, \qquad X_1 = \tfrac{1}{\sqrt{2}}\,e^{i(w\tau + m\sigma)}, \qquad X_2 = \tfrac{1}{\sqrt{2}}\,e^{i(w\tau - m\sigma)}, \qquad \kappa^2 \equiv \mu^2 = w^2 + m^2. \tag{D.4}$$

In the reduced theory we have $\mu^2\cos 2\varphi = \partial_+X_i\partial_-X_i^*$, so that the corresponding solution has $\varphi = \varphi_0 = \mathrm{const}$, with

$$\cos 2\varphi_0 = \frac{w^2 - m^2}{w^2 + m^2}, \qquad \sin\varphi_0 = \frac{m}{\mu}, \qquad \cos\varphi_0 = \frac{w}{\mu}. \tag{D.5}$$

[21] This Lagrangian is found by starting with the reduced theory based on the symmetrically gauged $G/H = SO(1,2)/SO(2) \times SO(3)/SO(2)$ gWZW model and integrating out the $SO(2) \times SO(2)$ gauge fields [1].

[22] More general solutions of CSG were discussed in [12, 13].

Also, for $\theta = n\sigma + a\tau$, the equation of motion for ϕ implies

$$\mu^2 \sin^4 \varphi_0 = n^2 - a^2, \qquad \text{i.e.} \quad \frac{m^4}{w^2 + m^2} = n^2 - a^2. \tag{D.6}$$

Note that here we cannot set $n = 0$. If σ is periodic n should be an integer, which imposes constraints on m and w. A special solution with $w = 0$ (i.e., $J = 0$) corresponds to $\varphi_0 = \frac{\pi}{2}$ and $m = n$.

The embedding of the circular string solution into the reduced model considered in (4.13) was different—it contained only 2D time dependence. Note that had we started with the axially gauged $SO(3)/SO(2)$ WZW model the $\cot^2 \varphi$ in the kinetic term would be replaced by $\tan^2 \varphi$ and the a^2 and n^2 terms in (D.6) would change places. In this case we could get a solution of the reduced theory which looks more like that found in (4.13).

Indeed, if we replace v by $e^{i\theta}$, $\frac{\omega}{\kappa}$ by $\cos\varphi$ and $\frac{m}{\kappa}$ by $\sin\varphi$ in (4.13) and then integrate out $A_\pm$ at a classical level, we get the complex sine-Gordon model with $\tan^2 \varphi$ in the kinetic term. This is also related to the fact that the point-like or BMN limit of the above solution ($m \to 0$) corresponds to the trivial vacuum in the reduced theory (see [3]), which was not the case in (4.13).[23]

For the homogeneous solution in $AdS_3 \times S^1$, (4.35), (i.e., the limit of large κ and large ℓ when we can ignore periodicity of σ), we have $\partial_+ Y_0 \partial_- Y_0 + \partial_+ Y_{-1} \partial_- Y_{-1} - \partial_+ Y_1 \partial_- Y_1 - \partial_+ Y_2 \partial_- Y_2 = \mu^2 \cosh 2\phi$ ($\mu = \nu$). Then

$$\kappa^2 + \ell^2 = \mu^2 \cosh 2\phi_0, \qquad \sinh \phi_0 = \frac{\ell}{\mu}, \qquad \cosh \phi_0 = \frac{\kappa}{\mu}, \qquad \mu = \sqrt{\kappa^2 - \ell^2}. \tag{D.7}$$

Thus the solution is

$$\phi = \phi_0, \qquad \chi = k\sigma + b\tau, \qquad k^2 - b^2 = \mu^2 \sinh^4 \phi_0, \qquad \sinh \phi_0 = \frac{\ell}{\mu}. \tag{D.8}$$

In the scaling limit k and m need not be integers. As long as we decompactify σ we can always rotate b to 0 by a 2D Lorentz transformation since this is a symmetry of the reduced theory. Also, k needs to be non-zero.

Starting with the axially gauged or 'T-dual' model with $\coth^2 \phi \to \tanh^2 \phi$ would interchange k and b. Again, the reduced theory embedding of the solution (4.35) discussed in (4.37), (4.38) was only τ-dependent and thus was different.

Appendix E. An alternative computation of reduced theory fluctuation frequencies

In section 4.3 we partially fixed the H gauge symmetry such that two of the physical fluctuation fields in $\eta^{\parallel}$ decoupled from the remaining unphysical fluctuation fields. However, this strategy may not necessarily work for other homogeneous solutions, e.g., the 'small' spinning string in $\mathbb{R}_t \times S^5$ discussed in [23, 24]. Here we shall use the example of the S^5 sector of the reduced theory solution corresponding to the two-spin homogeneous string in $\mathbb{R}_t \times S^3$ (section 4.3.1), to discuss an alternative strategy for computing the characteristic frequencies.

We introduce the following parametrization of $\eta^{\parallel}$, $\eta^{\perp}$ and $\delta A_\pm$,

$$\eta^{\parallel} = \begin{pmatrix} 0 & 0 & b_1 + ib_2 & b_3 + ib_4 \\ 0 & 0 & -b_3 + ib_4 & b_1 - ib_2 \\ -b_1 + ib_2 & b_3 + ib_4 & 0 & 0 \\ -b_3 + ib_4 & -b_1 - ib_2 & 0 & 0 \end{pmatrix}, \tag{E.1}$$

[23] It should be noted that here the gauge group, $SO(2)$, is Abelian and thus the axial gauging, $\tau(u) = -u$, is allowed. For non-Abelian groups this is not possible as such a map τ is no longer an automorphism of the algebra. Instead, we may use automorphisms like those discussed in section 2.2.

$$\eta^{\perp} = \begin{pmatrix} \mathrm{i}h_1 & h_2 + \mathrm{i}h_3 & 0 & 0 \\ -h_2 + \mathrm{i}h_3 & -\mathrm{i}h_1 & 0 & 0 \\ 0 & 0 & \mathrm{i}h_4 & h_5 + \mathrm{i}h_6 \\ 0 & 0 & -h_5 + \mathrm{i}h_6 & -\mathrm{i}h_4 \end{pmatrix}, \tag{E.2}$$

$$\delta A_+ = \begin{pmatrix} \mathrm{i}a_{+1} & 0 & 0 & 0 \\ 0 & -\mathrm{i}a_{+1} & 0 & 0 \\ 0 & 0 & \mathrm{i}a_{+4} & 0 \\ 0 & 0 & 0 & -\mathrm{i}a_{+4} \end{pmatrix},$$
$$\delta A_- = \begin{pmatrix} \mathrm{i}a_{-1} & a_{-2} + \mathrm{i}a_{-3} & 0 & 0 \\ -a_{-2} + \mathrm{i}a_{-3} & -\mathrm{i}a_{-1} & 0 & 0 \\ 0 & 0 & \mathrm{i}a_{-4} & a_{-5} + \mathrm{i}a_{-6} \\ 0 & 0 & -a_{-5} + \mathrm{i}a_{-6} & -\mathrm{i}a_{-4} \end{pmatrix}. \tag{E.3}$$

Using the H gauge freedom we set the off-diagonal components of δA_+ to zero. When we substitute these expressions into the bosonic part of the quadratic fluctuation Lagrangian (4.3), we get a Lagrangian with constant coefficients. As in section 4.3.1 the fields decouple into two smaller sectors, the first sector containing b_3, b_4 and the diagonal components of $\eta^{\perp}$, $\delta A_{\pm}$, and the second sector containing b_1,b_2 and the off-diagonal components of $\eta^{\perp}$, $\delta A_{\pm}$.

We can easily integrate out the diagonal components of $\delta A_{\pm}$, and then end up with a Lagrangian for 14 fields (4 of $\eta^{\parallel}$, 6 of $\eta^{\perp}$, 4 of δA_-), some of which are unphysical. Using the fact that we have the two decoupled sectors, we can split the corresponding 14×14 mass matrix into two parts, a 4×4 matrix containing b_3 and b_4, and a 10×10 matrix containing b_1 and b_2.

Substituting $\mathrm{e}^{\mathrm{i}(\Omega\tau - n\sigma)}$ into the equations of motion we find that the 4×4 matrix takes the form

$$\begin{pmatrix} 4(n^2-\Omega^2) & 8\mathrm{i}\kappa\Omega & -\frac{2m(n^2-\Omega^2)}{\sqrt{\kappa^2-m^2}} & -\frac{2m(n^2-\Omega^2)}{\sqrt{\kappa^2-m^2}} \\ -8\mathrm{i}\kappa\Omega & -4(4m^2-n^2+\Omega^2) & \frac{4\mathrm{i}\kappa m\Omega}{\sqrt{\kappa^2-m^2}} & \frac{4\mathrm{i}\kappa m\Omega}{\sqrt{\kappa^2-m^2}} \\ \frac{2m\sqrt{\kappa^2-m^2}(n^2-\Omega^2)}{-\kappa^2+m^2} & -\frac{4\mathrm{i}\kappa m\Omega}{\sqrt{\kappa^2-m^2}} & -\frac{m^2(n^2-\Omega^2)}{-\kappa^2+m^2} & -\frac{m^2(n^2-\Omega^2)}{-\kappa^2+m^2} \\ \frac{2m\sqrt{\kappa^2-m^2}(n^2-\Omega^2)}{-\kappa^2+m^2} & -\frac{4\mathrm{i}\kappa m\Omega}{\sqrt{\kappa^2-m^2}} & -\frac{m^2(n^2-\Omega^2)}{-\kappa^2+m^2} & -\frac{m^2(n^2-\Omega^2)}{-\kappa^2+m^2} \end{pmatrix}. \tag{E.4}$$

This matrix has rank 2, i.e. it has two non-vanishing eigenvalues. The resulting two characteristic frequencies are then found to be the same as those in (4.26) in section 4.3.1,

$$\sqrt{n^2 + 2\kappa^2 - 2m^2 \pm 2\sqrt{n^2\kappa^2 + (m^2-\kappa^2)^2}}, \qquad n \in \mathbb{Z}. \tag{E.5}$$

The 10×10 matrix has rank 10. The condition that its determinant vanishes gives the following characteristic fluctuation frequencies,

$$2 \times \sqrt{n^2 + \kappa^2 - 2m^2}, \qquad n \in \mathbb{Z}, \tag{E.6}$$

and

$$2 \times n \pm \kappa, \qquad 2 \times n \pm \frac{\kappa^2 - 2m^2}{\kappa}. \qquad n \in \mathbb{Z}. \tag{E.7}$$

The frequencies in (E.6) are the same as those in (4.25) in section 4.3.1 (i.e., like (E.5) they match the frequencies found from the conformal-gauge string theory).

The frequencies in (E.7) give a trivial (κ, m-independent) contribution to the one-loop partition function that should be cancelled against ghost (or path integral measure) terms.

The approach employed here, i.e. evaluating a larger mass matrix including unphysical fluctuations in addition to physical fluctuations, should also be applicable to other homogeneous solutions. In particular, we can apply it to the homogeneous string solution discussed in section 4.3.2. However, it is not clear whether it may be useful for extending the computation to the two-loop level as the unphysical modes (which we did not explicitly decouple above, as this was irrelevant at the one-loop level), may get coupled through the interaction terms.

References

[1] Grigoriev M and Tseytlin A A 2008 Pohlmeyer reduction of $AdS_5 \times S^5$ superstring sigma model *Nucl. Phys.* B **800** 450 (arXiv:0711.0155 [hep-th])

[2] Mikhailov A and Schafer-Nameki S 2008 Sine-Gordon-like action for the superstring in $AdS_5 \times S^5$ *J. High Energy Phys.* JHEP05(2008)075 (arXiv:0711.0195 [hep-th])

[3] Grigoriev M and Tseytlin A A 2008 On reduced models for superstrings on $AdS_n \times S^n$ *Int. J. Mod. Phys.* A **23** 2107 (arXiv:0806.2623 [hep-th])

[4] Roiban R and Tseytlin A A 2009 UV finiteness of Pohlmeyer-reduced form of the $AdS_5 \times S^5$ superstring theory *J. High Energy Phys.* JHEP04(2009)078 (arXiv:0902.2489 [hep-th])

[5] Pohlmeyer K 1976 Integrable Hamiltonian systems and interactions through quadratic constraints *Commun. Math. Phys.* **46** 207

[6] Tseytlin A A 2004 Spinning strings and AdS/CFT duality *From Fields to Stings: Circumnavigating Theoretical Physics* vol 2 ed M Shifman, A Vainshtein and J Wheater (Singapore: World Scientific) 1648–1707 (arXiv:hep-th/0311139)

[7] Mikhailov A 2006 An action variable of the sine-Gordon model *J. Geom. Phys.* **56** 2429 (arXiv:hep-th/0504035)

[8] Mikhailov A 2005 A nonlocal Poisson bracket of the sine-Gordon model arXiv:hep-th/0511069

[9] De Vega H J and Sanchez N G 1993 Exact integrability of strings in D-dimensional De Sitter space-time *Phys. Rev.* D **47** 3394

Combes F, de Vega H J, Mikhailov A V and Sanchez N G 1994 Multistring solutions by soliton methods in de Sitter space-time *Phys. Rev.* D **50** 2754 (arXiv:hep-th/9310073)

[10] Hofman D M and Maldacena J M 2006 Giant magnons *J. Phys. A: Math. Gen.* **39** 13095 (arXiv:hep-th/0604135)

[11] Chen H Y, Dorey N and Okamura K 2006 Dyonic giant magnons *J. High Energy Phys.* JHEP09(2006)024 (arXiv:hep-th/0605155)

[12] Okamura K and Suzuki R 2007 A perspective on classical strings from complex sine-Gordon solitons *Phys. Rev.* D **75** 046001 (arXiv:hep-th/0609026)

[13] Hayashi H, Okamura K, Suzuki R and Vicedo B 2007 Large winding sector of AdS/CFT *J. High Energy Phys.* JHEP11(2007)033 (arXiv:0709.4033 [hep-th])

[14] Klose T and McLoughlin T 2008 Interacting finite-size magnons *J. Phys. A: Math. Theor.* **41** 285401 (arXiv:0803.2324 [hep-th])

[15] Jevicki A, Jin K, Kalousios C and Volovich A 2008 Generating AdS string solutions *J. High Energy Phys.* JHEP03(2008)032 (arXiv:0712.1193 [hep-th])

Jevicki A and Jin K 2008 Solitons and AdS string solutions *Int. J. Mod. Phys.* A **23** 2289 (arXiv:0804.0412 [hep-th])

Aniceto I and Jevicki A 2008 *N*-body dynamics of giant magnons in $\mathbb{R} \times S^2$ arXiv:0810.4548 [hep-th]

Jevicki A and Jin K 2009 Moduli dynamics of AdS_3 strings arXiv:0903.3389 [hep-th]

[16] Miramontes J L 2008 Pohlmeyer reduction revisited *J. High Energy Phys.* JHEP10(2008)087 (arXiv:0808.3365 [hep-th])

Hollowood T J and Miramontes J L 2009 Magnons, their solitonic avatars and the Pohlmeyer reduction *J. High Energy Phys.* JHEP04(2009)060 (arXiv:0902.2405 [hep-th])

Hollowood T J and Miramontes J L 2009 A new and elementary $\mathbb{C}\sharp P^n$ dyonic magnon arXiv:0905.2534 [hep-th]

[17] Dorn H, Jorjadze G and Wuttke S 2009 On spacelike and timelike minimal surfaces in AdS_n arXiv:0903.0977 [hep-th]

Giombi S, Pestun V and Ricci R 2009 Notes on supersymmetric Wilson loops on a two-sphere arXiv:0905.0665 [hep-th]

[18] Metsaev R R and Tseytlin A A 1998 Type IIB superstring action in $AdS_5 \times S^5$ background *Nucl. Phys.* B **533** 109 (arXiv:hep-th/9805028)

[19] Roiban R, Tirziu A and Tseytlin A A 2007 Two-loop world-sheet corrections in $AdS_5 \times S^5$superstring *J. High Energy Phys.* JHEP07(2007)056 (arXiv:0704.3638 [hep-th])

[20] Roiban R and Tseytlin A A 2007 Strong-coupling expansion of cusp anomaly from quantum superstring *J. High Energy Phys.* JHEP11(2007)016 (arXiv:0709.0681 [hep-th])

Roiban R and Tseytlin A A 2008 Spinning superstrings at two loops: strong-coupling corrections to dimensions of large-twist SYM operators *Phys. Rev.* D **77** 066006 (arXiv:0712.2479 [hep-th])

[21] Arutyunov G and Frolov S 2009 Foundations of the $AdS_5 \times S^5$ superstring. Part I arXiv:0901.4937 [hep-th]

[22] Fradkin E S and Tseytlin A A 1985 Quantum equivalence of dual field theories *Ann. Phys.* **162** 31

[23] Frolov S and Tseytlin A A 2003 Multi-spin string solutions in $AdS_5 \times S^5$ *Nucl. Phys.* B **668** 77 (arXiv:hep-th/0304255)

[24] Frolov S and Tseytlin A A 2003 Quantizing three-spin string solution in $AdS_5 \times S^5$ *J. High Energy Phys.* JHEP07(2003)016 (arXiv:hep-th/0306130)

[25] Arutyunov G, Russo J and Tseytlin A A 2004 Spinning strings in $AdS_5 \times S^5$: new integrable system relations *Phys. Rev.* D **69** 086009 (arXiv:hep-th/0311004)

[26] Frolov S A, Park I Y and Tseytlin A A 2005 On one-loop correction to energy of spinning strings in S^5 *Phys. Rev.* D **71** 026006 (arXiv:hep-th/0408187)

[27] Park I Y, Tirziu A and Tseytlin A A 2005 Spinning strings in $AdS_5 \times S^5$: one-loop correction to energy in $SL(2)$sector *J. High Energy Phys.* JHEP03(2005)013 (arXiv:hep-th/0501203)

[28] Beisert N, Tseytlin A A and Zarembo K 2005 Matching quantum strings to quantum spins: one-loop versus finite-size corrections *Nucl. Phys.* B **715** 190 (arXiv:hep-th/0502173)

[29] Gubser S S, Klebanov I R and Polyakov A M 2002 A semi-classical limit of the gauge/string correspondence *Nucl. Phys.* B **636** 99 (arXiv:hep-th/0204051)

[30] Frolov S and Tseytlin A A 2002 Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$ *J. High Energy Phys.* JHEP06(2002)007 (arXiv:hep-th/0204226)

[31] Frolov S, Tirziu A and Tseytlin A A 2007 Logarithmic corrections to higher twist scaling at strong coupling from AdS/CFT *Nucl. Phys.* B **766** 232 (arXiv:hep-th/0611269)

[32] Beisert N and Freyhult L 2005 Fluctuations and energy shifts in the Bethe ansatz *Phys. Lett.* B **622** 343 (arXiv:hep-th/0506243)

Gromov N and Vieira P 2008 The $AdS_5 \times S^5$ superstring quantum spectrum from the algebraic curve *Nucl. Phys.* B **789** 175 (arXiv:hep-th/0703191)

[33] Papathanasiou G and Spradlin M 2007 Semiclassical quantization of the giant magnon *J. High Energy Phys.* JHEP06(2007)032 (arXiv:0704.2389 [hep-th])

[34] Minahan J A 2003 Circular semiclassical string solutions on $AdS_5 \times S^5$ *Nucl. Phys.* B **648** 203 (arXiv:hep-th/0209047)

[35] Arutyunov G, Frolov S and Zamaklar M 2007 Finite-size effects from giant magnons *Nucl. Phys.* B **778** 1 (arXiv:hep-th/0606126)

[36] Kruczenski M and Tseytlin A A 2004 Semiclassical relativistic strings in $S^5$and long coherent operators in $\mathcal{N} = 4$ SYM theory *J. High Energy Phys.* JHEP09(2004)038 (arXiv:hep-th/0406189)

[37] Whittaker E T and Watson G N 1958 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)

[38] Pawellek M 2009 Quantization of sine-Gordon solitons on the circle: semiclassical versus exact results *Nucl. Phys.* B **810** 527 (arXiv:0808.0696 [hep-th])

[39] Correa F, Dunne G V and Plyushchay M S 2009 The Bogoliubov/de Gennes system, the AKNS hierarchy, and nonlinear quantum mechanical supersymmetry arXiv:0904.2768 [hep-th]

Dunne G V 2008 Functional determinants in quantum field theory *J. Phys. A: Math. Theor.* **41** 304006 (arXiv:0711.1178 [hep-th])

Braden H W 1985 Periodic functional determinants *J. Phys. A: Math. Gen.* **18** 2127